論 說 報 告

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THE APPLICATION OF THE THEORY OF INFLUENCE EQUATIONS FOR THE ANALYSIS OF SECONDARY STRESSES IN TRUSS BRIDGES.

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Synopsis.

This paper presents the application of the Theory of Influence Equations, the derivation of which was introduced by the writer in the September 1932 journal, to the analysis of secondary stresses in truss bridges.

INTRODUCTION.

The primary stresses in members of a bridge analized on the assumption that the ends of each member are perfectly free to rotate around the pannel points and gravity axes of members coincide with the center lines of the members.

Actually the members are connected with rivets at pannel points or pins. These pins are not frictionless and offer considerable resistance to the turning of members at the joints. The gravity axes may not coincide with the center lines of members, and the members may not be straight. Horizontal and inclined members act as beam to carry their own weight or intervening loads applied on members between joints. A temperature change may cause a difference in length of members.

Due to all these causes the members are subjected to bending moments and shears besides axial stresses. These additional stresses are known as secondary stresses.

The analysis of secondary stresses can also be done by means of the Influence Equation Theory.

ANALYSIS.

Take truss as shown in **Fig. 1.**

From the deflection moment equations:

$$AM_1=-(2m_1+m_2)-A\alpha_{ab} \qquad CM_6=-(2m_6+m_5)+C\beta_{ca}$$
$$AM_2=-(2m_2+m_1)+A\beta_{ab} \qquad DM_7=-(2m_7+m_8)-D\alpha_{be}$$
$$BM_3=-(2m_3+m_4)-B\alpha_{bc} \qquad DM_8=-(2m_8+m_7)+D\beta_{be}$$
$$BM_4=-(2m_4+m_3)+B\beta_{bc} \qquad EM_9=-(2m_9+m_{10})-E\alpha_{ec}$$
$$CM_5=-(2m_5+m_6)-C\alpha_{ca} \qquad EM_{10}=-(2m_{10}+m_9)+E\beta_{ec}$$

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$FM_{11}=-(2m_{11}+m_{12})-F\alpha_{bd}$
$FM_{12}=-(2m_{12}+m_{11})+F\beta_{bd}$
$GM_{13}=-(2m_{13}+m_{14})-G\alpha_{de}$
$GM_{14}=-(2m_{14}+m_{13})+G\beta_{de}$
$HM_{15}=-(2m_{15}+m_{16})-H\alpha_{df}$
$HM_{16}=-(2m_{16}+m_{15})+H\beta_{df}$
$IM_{17}=-(2m_{17}+m_{18})-I\alpha_{fe}$
$IM_{18}=-(2m_{18}+m_{17})+I\beta_{fe}$
$JM_{19}=-(2m_{19}+m_{20})-J\alpha_{fg}$
$JM_{20}=-(2m_{20}+m_{19})+J\beta_{fg}$
$KM_{21}=-(2m_{21}+m_{22})-K\alpha_{ge}$
$KM_{22}=-(2m_{22}+m_{21})+K\beta_{ge}$

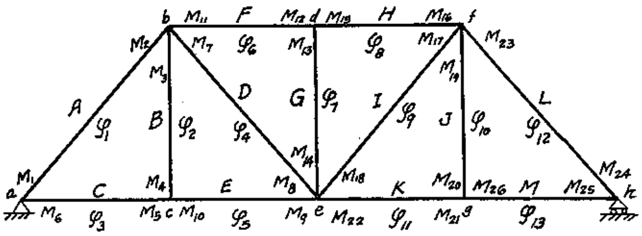

Fig. 1.

$LM_{23}=-(2m_{23}+m_{24})-L\alpha_{fh}$
$LM_{24}=-(2m_{24}+m_{23})+L\beta_{fh}$
$MM_{25}=-(2m_{25}+m_{26})-M\alpha_{hg}$
$MM_{26}=-(2m_{26}+m_{25})+M\beta_{hg}$

From angular relations:

$m_6=m_1-\varphi_1+\varphi_3$ $\quad m_{12}=m_{13}-\varphi_7+\varphi_6$ $\quad m_{16}=m_{19}-\varphi_{10}+\varphi_8$
$m_2=m_3-\varphi_2+\varphi_1$ $\quad m_{15}=m_{13}-\varphi_7+\varphi_8$ $\quad m_{17}=m_{19}-\varphi_{10}+\varphi_9$
$m_7=m_3-\varphi_2+\varphi_4$ $\quad m_9=m_{14}-\varphi_7+\varphi_5$ $\quad m_{23}=m_{19}-\varphi_{10}+\varphi_{12}$
$m_{11}=m_3-\varphi_2+\varphi_6$ $\quad m_8=m_{14}-\varphi_7+\varphi_4$ $\quad m_{21}=m_{20}-\varphi_{10}+\varphi_{11}$
$m_5=m_4-\varphi_2+\varphi_3$ $\quad m_{18}=m_{14}-\varphi_7+\varphi_9$ $\quad m_{26}=m_{20}-\varphi_{10}+\varphi_{13}$
$m_{10}=m_4-\varphi_2+\varphi_5$ $\quad m_{22}=m_{14}-\varphi_7+\varphi_{11}$ $\quad m_{25}=m_{24}-\varphi_{12}+\varphi_{13}$

Substituting the angular relations into moment equations and arranging the equations into tabulated form and eliminating "m" we have,

Table 1.

Eq.	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}	M_{12}	M_{13}	M_{14}	M_{15}	M_{16}	M_{17}	M_{18}	M_{19}	M_{20}	M_{21}	M_{22}	M_{23}	M_{24}	M_{25}	M_{26}
1	A	-2A	2B	-B																							
2			B	-2B	2C	-C																				
3	2A	-A			C	-2C																				
4			-2B	B			2D	-D																		
5							D	-2D	2E	-E																
6			-B	2B					E	-2E																
7							-2D	D			2F	-F														
8											F	-2F	2G	-G												
9							-D	2D					G	-2G												
10													-2G	G	2H	-H										
11															H	-2H	2I	-I								
12													-G	2G			I	-2I								
13																	-2I	I	2J	-J						
14																			J	-2J	2K	-K				
15																	-I	2I			K	-2K				
16																			-2J	J			2L	-L		
17																							L	-2L	2M	-M
18																			-J	2J					M	-2M
19	1					1																				
20		1	1				1				1															
21				1	1					1																	
22												1	1		1												
23								1	1					1				1				1				
24																1	1		1				1			
25																				1	1					1
26																								1	1	

Eq.	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	α_{ab}	β_{ab}	α_{bc}	β_{bc}	α_{ca}	β_{ca}	α_{be}	β_{be}	α_{ec}	β_{ec}	α_{bd}	β_{bd}	α_{de}	β_{de}	α_{df}	β_{df}	α_{fe}	β_{fe}	α_{fg}	β_{fg}	α_{ge}	β_{ge}	α_{fh}	β_{fh}	α_{hg}	β_{hg}
1	-3	3												A	2A	2B	B																						
2		-3	3													B	2B	2C	C																				
3	3		-3										2A	A			C	2C																				
4		-3		3													-2B	-B			2D	D																		
5				-3	3																D	2D	2E	E																
6		3			-3												-B	-2B					E	2E																
7				-3		3															-2D	-D			2F	F														
8						-3	3																	F	2F	2G	G												
9				3			-3															-D	-2D					G	2G												
10							-3	3																				-2G	-G	2H	H										
11								-3	3																					H	2H	2I	I								
12							3		-3																			-G	-2G			I	2I								
13									-3	3																								-2I	-I	2J	J						
14										-3	3																									J	2J	2K	K				
15									3		-3																							-I	-2I			K	2K				
16										-3		3																									-2J	-J			2L	L		
17												-3	3																											L	2L	2M	M
18										3			-3																							-J	-2J			M	2M
19																																								
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If we examine the arrangement of Relational Moment Equations in the Influence Equations it will be found that the terms of moments are successively arranged in each triangle. Therefore the Influence Equations for any truss may be written by inspection.

In the equations we have terms of deflection angle "φ" which in this case are treated as known values and are determined independently.

The values of "φ" change according to the loading on the bridge but relation in the Influence Equations remain the same, therefore, the Influence Values of each moment are determined by changing the value of "φ" according to the loading.

Since we now have 26 unknowns and 26 conditional equations the problem is capable of solution.

If in the analysis there are no intervening loads applied between joints or if the own weight, acting as loads on the beam, of inclined and horizontal members are disregarded then the terms of Constants, α and β, in the equations disappear. If the own weights of the inclined and horizontal members are taken into consideration then the values of the constants will be $\alpha=\beta=\frac{w}{12}l^2$,

Where w=intensity of uniform load due to own weight of member which is acting perpendicularly to the axis of the member.

l=length of the member.

If symmetrical loads are applied on the bridge then the number of unknowns and conditional equations are reduced to almost half.

After establishing the Influence Equations the values of "φ" must be determined.

The rigid triangle a-b-c, **Fig. 2**, is subject unit stresses S_{ab}, S_{bc}, S_{ca} and the length of the corresponding members are l_{ab}, l_{bc}, l_{ca}.

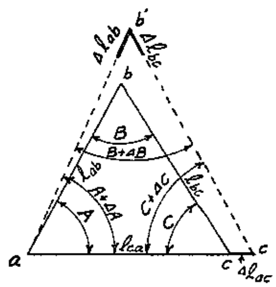

Fig. 2.

The change in length of each member will be

$$\Delta l_{ab}=(S_{ab}\times l_{ab})\div E \quad \cdots\cdots (1)$$

$$\Delta l_{bc}=(S_{bc}\times l_{bc})\div E \quad \cdots\cdots (2)$$

$$\Delta l_{ca}=(S_{ca}\times l_{ca})\div E \quad \cdots\cdots (3)$$

The angular change due to the change of length of members will be derived by applying the law of Sines.

$$(l_{bc}+\Delta l_{bc}):(l_{ca}+\Delta l_{ca})::\sin(A+\Delta A):\sin(B+\Delta B) \quad \cdots\cdots (4)$$

$$(l_{ca}+\Delta l_{ca}):(l_{ab}+\Delta l_{ab})::\sin(B+\Delta B):\sin(C+\Delta C) \quad \cdots\cdots (5)$$

$$\Delta A+\Delta B+\Delta C=0 \quad \cdots\cdots (6)$$

Expanding above equations and substituting the linear changes, we get

$$E\Delta A=(S_{bc}-S_{ab})\cot B+(S_{bc}-S_{ca})\cot C \quad \cdots\cdots (7)$$

$$E\Delta B=(S_{ca}-S_{bc})\cot C+(S_{ca}-S_{ab})\cot A \quad \cdots\cdots (8)$$

$$E\Delta C=(S_{ab}-S_{ca})\cot A+(S_{ab}-S_{bc})\cot B \quad \cdots\cdots (9)$$

These angular changes do not represent the values of "φ" directly, therefore the relation between the angular change and deflection angles "φ" must be established.

Fig. 3 shows four typical and possible cases which may be encountered. The full and dotted lines in the sketch represent the position of the axes of members before and after the change of angle, respectively.

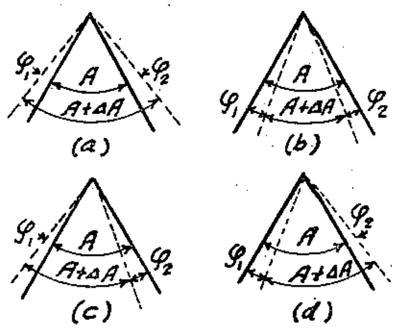

Fig. 3.

(a) From the geometrical relation, $A+\Delta A-\varphi_1-\varphi_2=A$, $\Delta A=\varphi_1+\varphi_2$ From the convention the sign of "φ_1" is positive and "φ_2" is negative. Therefore $\Delta A=\varphi_1-\varphi_2$

(b) In the similar manner, we have $\Delta A+A+\varphi_1+\varphi_2=A$, $\Delta A=-\varphi_1-\varphi_2$, $\varphi_1(-)$, $\varphi_2(+)$
$\therefore\ \Delta A=\varphi_1-\varphi_2$

(c) $A+\Delta A-\varphi_1+\varphi_2=A$, $\Delta A=\varphi_1-\varphi_2$, $\varphi_1(+)$, $\varphi_2(+)$ $\quad\therefore\ \Delta A=\varphi_1-\varphi_2$

(d) $A+\Delta A+\varphi_1-\varphi_2=A$, $\Delta A=-\varphi_1+\varphi_2$, $\varphi_1(-)$, $\varphi_2(-)$ $\quad\therefore\ \Delta A=\varphi_1-\varphi_2$

Generally the relation between the angular change and the deflection angles "φ" can be expressed thus:

$$\Delta A=\varphi_1-\varphi_2 \qquad\qquad (10)$$

If we examine the Influence Equations in **table 1** it will be found that the "φ" part in (6) equation can be expressed in terms of the angular change ΔA, ΔB etc.

The member ratio, $A=\dfrac{l}{2EI}$ in the Influence Equations can be simplified by multiplying "$2E$" throughout the equation. Then the Member Ratio become simpler thus: $A=\dfrac{l}{I}$ etc. and the coefficient for $\varphi_1-\varphi_2$ become $6E\Delta A$.

This will be illustrated by the following problem.

Illustrative Problem. Establish Influence Equations for the structure shown in **Fig. 4** neglecting the effects of the own weights of members and find the moments of both ends of members when the structure has the following properties.

The length of members:

$l_{ab}=l_{bc}=l_{cd}=l_{de}=25$ ft.

$l_{ac}=l_{ce}=l_{bd}=30$ ft.

The moment of inertia of the members:

$I_{ab}=I_{bd}=I_{de}=360^{\text{in}^4}$. $\qquad I_{bc}=I_{cd}=60^{\text{in}^4}$

$I_{ac}=I_{ce}=40^{\text{in}^4}$.

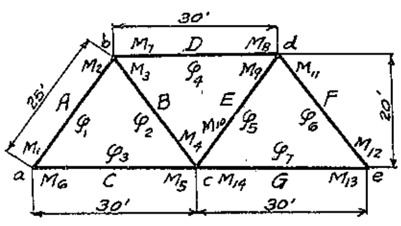

Fig. 4.

The unit stresses in the members:

$S_{ab}=S_{de}=-7\,000$ lbs. per sq. in.

$S_{ac}=S_{ce}=7\,300$ " " " "

$S_{bd}=-7\,330$ " " " "

$S_{bc}=S_{cd}=8\,000$ " " " "

Positive for tension, Negative for compression.

(a) By inspection, the Influence Equations can be written as shown in **table 2.** In this case the own weight of members are neglected therefore the terms of Constants, α and β, disappear from the equations.

Table 2.

	Eq.	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}	M_{12}	M_{13}	M_{14}	$\frac{E}{\Delta abc}$	$\frac{E}{\Delta bca}$	$\frac{E}{\Delta cab}$	$\frac{E}{\Delta cbd}$	$\frac{E}{\Delta bdc}$	$\frac{E}{\Delta dcb}$	$\frac{E}{\Delta cde}$	$\frac{E}{\Delta dec}$	$\frac{E}{\Delta ecd}$
Relational Moment Eqs.	1	-A	2A	-2B	B											6								
	2			-B	2B	-2C	C										6							
	3	-2A	A			-C	2C											6						
	4			2B	-B			-2D	D										6					
	5							-D	2D	-2E	E									6				
	6			B	-2B					-E	2E										6			
	7									2E	-E	-2F	F									6		
	8											-F	2F	-2G	G								6	
	9									E	-2E			-G	2G									6
ΣM=0 at Jt.	10	1					1																	
	11		1	1				1																
	12				1	1					1				1									
	13								1	1		1												
	14												1	1										

(b) In this case the structure and the unit stresses of members are symmetrical about the center line of the structure therefore the Influence Equations can be simplified.

$$M_8=-M_7,\ M_9=-M_3,\ M_{10}=-M_4,\ M_{11}=-M_2,\ M_{12}=-M_1,\ M_{13}=-M_6,\ M_{14}=-M_5,$$

$$\varphi_5=-\varphi_2,\ \varphi_6=-\varphi_1,\ \varphi_7=-\varphi_3,\ E=B,\ F=A,\ G=C.$$

Substituting these values into equations in **table 2** we have

Table 3.

	Eq.	M_1	M_2	M_3	M_4	M_5	M_6	M_7	$\frac{E}{\Delta abc}$	$\frac{E}{\Delta bca}$	$\frac{E}{\Delta cab}$	$\frac{E}{\Delta cbd}$	$\frac{E}{\Delta dcb}$
R. M. E.	1	-A	2A	-2B	B				6				
	2			-B	2B	-2C	C			6			
	3	-2A	A			-C	2C				6		
	4			2B	-B			-3D				6	
	5			2B	-4B								6
ΣM=0	6	1					1						
	7		1	1				1					

$\cot\angle abc=\cot\angle bcd=0.2915 \quad \cot\angle bac=\cot\angle bca=\cot\angle cbd=0.75$

$E\triangle abc=(7\,300-8\,000)\,0.75+(7\,300+7\,000)\,0.75=+10\,200$

Similarly,

$E\triangle bca=-15\,120 \qquad E\triangle cbd=+11\,500$

$E\triangle cab=+4\,875 \qquad E\triangle dcb=-23\,000$

Substituting the numerical values into equations in **table 3** and solving each moment, we have

Table 4.

Eq.	M_1	M_2	M_3	M_4	M_5	M_5	M_7	$\frac{E}{\Delta abc}$	$\frac{E}{\Delta bca}$	$\frac{E}{\Delta cab}$	$\frac{E}{\Delta cbd}$	$\frac{E}{\Delta dcb}$
1	-.8333	1.666	-10.000	5.000				6				
2			-5.000	10.000	-18.000	9.000			6			
3	-1.666	.833			-9.000	18.000				6		
4			10.000	-5.000							6	
5			10.000	-20.000								6
6	1					1						
7		1	1				1					
8	1							.0876	.2000	-.4000	.0626	.1064
9		1						1.5740	.0400	-.0800	1.1390	.1350
10			1					-.4550	-.0114	.0229	.2460	-.1815
11				1				-.2275	-.0057	.0115	.1230	-.3908
12					1			-.0438	-.4333	.2000	-.0313	-.2200
13						1		-.0876	-.2000	.4000	-.0626	-.1064
14							1	-1.1390	-.0286	.0571	-1.3850	.0465

In solving the problem numerically it is quicker to substituting the numerical values of $E\triangle abc$, $E\triangle bca$ etc. directly instead of the coefficient 6. However, in a truss with many more pannels any special member may have a greater stress under different loading; in that case the unit stress differ from the other loading.

Naturally the value of $E\triangle abc$, $E\triangle bca$ etc. changes but in case only the coefficient is used in the successive elimination of the unknowns then the process of elimination need not be repeated. The coefficients of $E\triangle abc$, $E\triangle bca$ etc. are the same for any symmetrical loading, therefore only the values of $E\triangle abc$, $E\triangle bca$ etc. need to be changed.

Multiplying the values of $E\triangle abc$, $E\triangle bca$ etc. by the coefficients in **table 4**, we have

$$M_1+0.0876\times 10\,200+0.2000\times(-15\,120)-0.4000\times 4\,875+0.0626\times 11\,500+0.1064\times(-23\,000)=0$$

$$M_1=+58\,000 \text{ in-lb.}$$

In the same manner,

$M_2=-25\,340$ in-lb. $M_3=-2\,630$ in-lb. $M_4=-8\,200$ in-lb.

$M_5=-11\,770$ in-lb. $M_6=-5\,800$ in-lb. $M_7=+27\,850$ in-lb.

The numerical example shown above is the same as the one given by Prof. M. S. Ketchum in his book "The Design of Steel Mill Building". The result of the analysis is the same as that obtained by Prof. Ketchum using different method.