論 說 報 告

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ON STRENGTH OF COLUMNS WITH VARIABLE CROSS SECTIONS

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Synopsis

The present paper covers the author's studies and results of tests on strength of columns with variable cross sections, such as columns with tapered, rhombic, parabolic flanges etc.

Contents

PART I. GENERAL CONSIDERATIONS.

1. Preface.

The object of this paper is to submit a method of solution of the elastic strength of the straight weightless columns with continuously or discontinuously varying moments of inertia, and to report the results of the theoretical solutions, numerical calculations, and also the results of experiments performed by the author.

On the similar problem, Bleich[1] has made some elaborated approximate calculations, while, after the author's opinion, the present method of considerations might bring more general and reliable results.

2. Fundamental Equations.

(a) Column with Continuously Varying Moment of Inertia.

Let $$y = AF_1(x) + BF_2(x) \quad \cdots\cdots (1)$$
where A and B are constants,
be the general solution of the well known equation

$$EI(x)\frac{dy^2}{dx^2} + Py = 0 \quad \cdots\cdots (2)$$

where y = virtual deflection of the column,
E = modulus of elasticity,
$I(x)$ = variable moment of inertia,
P = centrally applied load.

Since, in equations (1) and (2), x may be referred to a certain origin, we may write the general expressions for y and y' at x_1 and x_2 as follows,

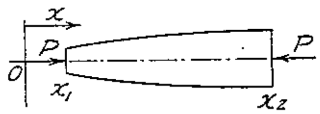

Fig. 1

$$y(x_1) = AF_1(x_1) + BF_2(x_1) \quad \cdots\cdots (3)$$
$$y(x_2) = AF_1(x_2) + BF_2(x_2) \quad \cdots\cdots (4)$$
$$y'(x_1) = AF_1'(x_1) + BF_2'(x_1) \quad \cdots\cdots (5)$$
$$y'(x_2) = AF_1'(x_2) + BF_2'(x_2) \quad \cdots\cdots (6)$$

Then, if we assume two hinges at x_1 and x_2, as shown in **Fig. 1**,

$$M(x_1) = 0, \quad M(x_2) = 0 \quad \cdots\cdots (7)$$

where $M(x)$ = virtual bending moment at x,

(1) Bleich—Theorie u. Berechung d. eiserne Brücken, 1924, pp. 134–142.

or $$y(x_1)=0, \quad y(x_2)=0 \quad \text{.......} (8)$$

Hence, by equations (3) and (4), as the stability condition of a column between x_1 and x_2, we shall have

$$\begin{vmatrix} F_1(x_1) & F_2(x_1) \\ F_1(x_2) & F_2(x_2) \end{vmatrix} = 0 \quad \text{.......} (9)$$

Consequently, by the roots of this equation we shall be able to determine the critical loads, which are such loads as they cause *no* or *any* deflection in the column. And we may assume that the least critical value P should be the primary critical load.

Similarly, the critical loads of a column hinged at x_1 and clamped at x_2 might be obtained from the following equation.

$$\begin{vmatrix} F_1(x_1) & F_2(x_1) \\ F_1'(x_2) & F_2'(x_2) \end{vmatrix} = 0 \quad \text{.......} (10)$$

and in a column both ends fixed,

$$\begin{vmatrix} F_1'(x_1) & F_2'(x_1) \\ F_1'(x_2) & F_2'(x_2) \end{vmatrix} = 0 \quad \text{.......} (11)$$

(b) Column with Discontinuously Varying Moment of Inertia.

When a column is hinged at both ends and has discontinuously varying moment of inertia as shown in **Fig. 2**, the following conditions should be satisfied.

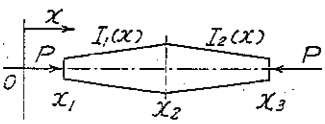

Fig. 2

$$\left.\begin{aligned} & y_1(x_1)=0 \\ & y_1(x_2)=y_2(x_2) \\ & y_1'(x_2)=y_2'(x_2) \\ \text{and} \quad & y_2(x_3)=0 \end{aligned}\right\} \quad \text{.......} (12)$$

where

$$EI_1(x)\,\frac{dy_1^2}{dx^2}+Py_1=0,$$

and $$EI_2(x)\,\frac{dy_2^2}{dx^2}+Py_2=0.$$

Or writing as

$$\left.\begin{aligned} & y_1=AF_1(x)+BF_2(x) \\ & y_2=C\Phi_1(x)+D\Phi_2(x) \\ \text{and} \quad & y_1'=AF_1'(x)+BF_2'(x) \\ & y_2'=C\Phi_1'(x)+D\Phi_2'(x) \end{aligned}\right\} \quad \text{.......} (13)$$

we·have, by eq. (12),

$$\begin{vmatrix} F_1(x_1) & F_2(x_1) & 0 & 0 \\ F_1(x_2) & F_2(x_2) & -\Phi_1(x_2) & -\Phi_2(x_2) \\ F_1'(x_2) & F_2'(x_2) & -\Phi_1'(x_2) & -\Phi_2'(x_2) \\ 0 & 0 & \Phi_1(x_3) & \Phi_2(x_3) \end{vmatrix} = 0 \quad \cdots\cdots\cdots(14)$$

In the present case, however, if the form of the column is symmetrical about x_2, the virtual deflection of the column should be also symmetrical due to the primary critical load.

Hence, for sush cases, we may simply put,

$$\left.\begin{array}{l} y_1(x_1)=0 \\ y_1'(x_2)=0 \end{array}\right\} \cdots\cdots\cdots(15)$$

or

$$\begin{vmatrix} F_1(x_1) & F_2(x_1) \\ F_1'(x_1) & F_2'(x_2) \end{vmatrix} = 0 \quad \cdots\cdots\cdots(16)$$

For the columns with more discontinuities of the moment of inertia or with different end conditions, we may generally extend the present principles for the case may be.

3. · Examples of Applications.

Before we go further on, it will not be useless to show some examples of application of the fundamental equations in the preceding article.

(a) Strength of a Triangular Plates.

When a triangular plate is loaded as shown in **Fig. 3**, the equation for the virtual deflection will be written as

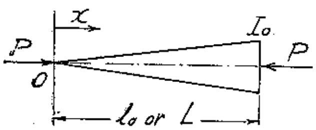

Fig. 3

$$\frac{EI_0}{l_0}x\frac{dy^2}{dx^2}+Py=0 \quad \cdots\cdots\cdots(17)$$

or putting, $\frac{Pl_0}{EI_0}=\lambda$, as

$$x\frac{dy^2}{dx^2}+\lambda y=0 \quad \cdots\cdots\cdots(18)$$

Solution of eq. (18) is, as we know,(1)

$$y=\sqrt{x}\,\{AJ_1(\sqrt{4\lambda x})+BN_1(\sqrt{4\lambda x})\} \quad \cdots\cdots\cdots(19)$$

where J and N denote Bessel's and Neumann's cylinder functions.

(1) Janke-Emde-Funktionentafeln, 1923, p. 167.

Consequently, a condition of the critical equilibrium of the plate should be

$$|J_1(\sqrt{4\lambda x_1})N_1(\sqrt{4\lambda L})-J_1(\sqrt{4\lambda L})N_1(\sqrt{4\lambda x_1})|_{x_1=0}=0 \quad \ldots\ldots \quad \ldots\ldots(20)$$

and knowing that $\left|\dfrac{J_1(\sqrt{4\lambda x_1})}{N_1(\sqrt{4\lambda x_1})}\right|_{x_1=0}=0$

and $N_1(\sqrt{4\lambda L})\neq\infty,$

eq. (20) will be rewritten as

$$J_1(\sqrt{4\lambda L})=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21)$$

Taking the least root of this equation, we have

$$\sqrt{4\lambda L}=3.83,$$

where $\lambda=\dfrac{PL}{EI_0}$.

Hence,

$$P=\frac{(3.83)^2}{4}\cdot\frac{EI_0}{L^2}=0.372\frac{\pi^2 EI_0}{L^2} \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

or we may know that the plate shall have about 37 % strength of the rectangular plate with a constant moment of inertia I_0.

(b) Strength of the Truncated Triangular Plate.

For a truncated triangular plate as shown in **Fig. 4**, the conditional equation for the critical stability will be given as follows.

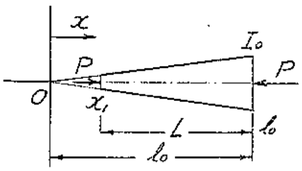

Fig. 4

Referring to eq. (19),

$$|J_1(\sqrt{4\lambda x_1})N_1(\sqrt{4\lambda l_0})-J_1(\sqrt{4\lambda l_0})N_1(\sqrt{4\lambda x_1})|_{x_1\neq0}=0 \quad \ldots\ldots(23)$$

As a numerical example, if we put

$$x_1=\frac{l_0}{4}$$

the least root of eq. (23) will be given as

$$\sqrt{4\lambda x_1}=3.20$$

and since $x_1=\dfrac{1}{3}L,\ \ l_0=\dfrac{4}{3}L$ and $\lambda=\dfrac{P}{EI_0}l_0=\dfrac{4P}{3EI_0}L,$

we have

$$P=\left(\frac{3}{4}\cdot\frac{3.20}{\pi}\right)^2\frac{EI_0}{L^2}=0.584\frac{\pi^2 EI_0}{L^2} \quad \ldots\ldots\ldots\ldots(24)$$

(c) Strength of the Rhombic Plate.

In the present case, let us consider a rhombic plate as shown in **Fig. 5**,

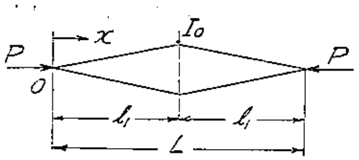

Fig. 5

Then, as the plate is symmetrical about $x=l_1$ axis, we may apply eq. (16), and knowing that for $0 \leqq x \leqq l_1$,

$$y=\sqrt{x}\,\{AJ_1(\sqrt{4\lambda x})+BN_1(\sqrt{4\lambda x})\},$$

where $\lambda=\dfrac{P}{EI_0}l_1$,

and

$$y'=\frac{1}{2\sqrt{x}}A\,\{J_1(\sqrt{4\lambda x})+\sqrt{4\lambda x}\,J_1'(\sqrt{4\lambda x})\}$$

$$+\frac{1}{2\sqrt{x}}B\,\{N_1(\sqrt{4\lambda x}+\sqrt{4\lambda x}\,N_1'(\sqrt{4\lambda x})\} \quad \ldots\ldots\ldots\ldots(25)$$

we have generally for the symmetrically truncated rhombic plate

$$\begin{vmatrix} J_1(\sqrt{4\lambda x_1}) & N_1(\sqrt{4\lambda x_1}) \\ J_1(\sqrt{4\lambda l_1})+\sqrt{4\lambda l_1}J_1'(\sqrt{4\lambda l_1}) & N_1(\sqrt{4\lambda l_1})+\sqrt{4\lambda l_1}N_1'(\sqrt{4\lambda l_1}) \end{vmatrix}=0\,..(26)$$

And putting $x_1=0$, we have

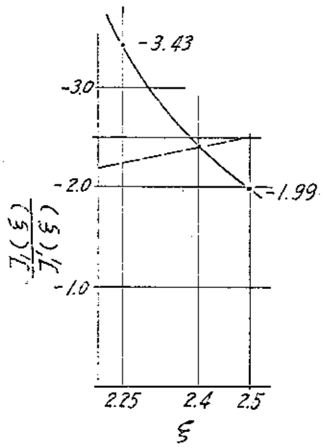

Fig. 6

$$J_1(\sqrt{4\lambda l_1})+\sqrt{4\lambda l_1}J_1'(\sqrt{4\lambda l_1})=0$$

or

$$\frac{J_1(\sqrt{4\lambda l_1})}{J_1'(\sqrt{4\lambda l_1})}=-\sqrt{4\lambda l_1}\ldots\ldots\ldots\ldots\ldots(27)$$

By the graphical representation, as shown in **Fig. 6**, the least root of eq. (27) will be given as

$$\sqrt{4\lambda l_1}=2.40,$$

where, $\lambda=\dfrac{P}{EI_0}l_1=\dfrac{P}{2EI_0}L.$

And we have

$$P=\left(\frac{2.40}{\pi}\right)^2\frac{\pi^2EI_0}{L^2}=0.584\frac{\pi^2EI_0}{L^2} \quad \ldots\ldots(28)$$

where

L=total length of the plate.

(d) Strength of the Plate with Two Different Moments of Inertia.

For the plate as shown in **Fig. 7**, as I_1 and I_2 are constants, we have

$$\left.\begin{aligned} y_1&=A\cos\lambda_1 x+B\sin\lambda_1 x \\ y_2&=C\cos\lambda_2 x+D\sin\lambda_2 x \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

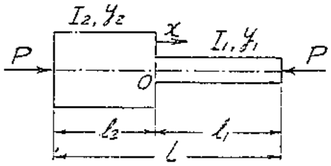

Fig. 7

where $\lambda_1=\sqrt{\dfrac{P}{EI_1}}$ and $\lambda_2=\sqrt{\dfrac{P}{EI_2}}$

and

$$\left.\begin{array}{l} y_1'=\lambda_1(-A\sin\lambda_1 x+B\cos\lambda_1 x) \\ y_2'=\lambda_2(-C\sin\lambda_2 x+D\cos\lambda_2 x) \end{array}\right\} \quad\cdots\cdots\cdots(30)$$

Therefore, by eq. (14), we have

$$\lambda_1\cos\lambda_1 l_1\sin\lambda_2 l_2+\lambda_2\sin\lambda_1 l_1\cos\lambda_2 l_2=0 \quad\cdots\cdots\cdots(31)^*$$

or

$$\frac{\tan\lambda_1 l_1}{\tan\lambda_2 l_2}=-\frac{\lambda_1}{\lambda_2}=-\sqrt{\frac{I_2}{I_1}} \quad\cdots\cdots\cdots(32)$$

As an example of solution of eq. (32), if we put

$$l_1=l_2=\frac{1}{2}L \quad\text{and}\quad I_2=2I_1,$$

we get

$$\frac{\tan\left(\dfrac{1}{\sqrt{2}}\lambda_2 L\right)}{\tan\left(\dfrac{1}{2}\lambda_2 L\right)}=-\sqrt{2} \quad\cdots\cdots\cdots(33)$$

Using the graphics as given in **Fig. 8**, we know that the root of eq. (33) will be

$$\frac{1}{\sqrt{2}}\lambda_2 L=1.79$$

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* Note on reduction of eq. (31).

Putting $x_1=-l_1$, $x_2=0$ and $x_3=+l_1$, we have

$F_1(x_1)=\cos\lambda_1 l_1$	$F_1(x_2)=1$
$F_2(x_1)=-\sin\lambda_1 l_1$	$F_2(x_2)=0$
$\Phi_1(x_2)=1$	$\Phi_1(x_3)=\cos\lambda_2 l_2$
$\Phi_2(x_2)=0$	$\Phi_2(x_3)=\sin\lambda_2 l_2$
$F_1'(x_2)=0$	$\Phi_1'(x_2)=0$
$F_2'(x_2)=\lambda_1$	$\Phi_2'(x_2)=\lambda_2$

Therefore by eq. (14),

$$\begin{vmatrix} \cos\lambda_1 l_1 & -\sin\lambda_1 l_1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & \lambda_1 & 0 & -\lambda_2 \\ 0 & 0 & \cos\lambda_2 l_2 & \sin\lambda_2 l_2 \end{vmatrix}=0$$

or

$$\cos\lambda_1 l_1\begin{vmatrix} 0 & -1 & 0 \\ \lambda_1 & 0 & -\lambda_2 \\ 0 & \cos\lambda_2 l_2 & \sin\lambda_2 l_2 \end{vmatrix}-1\begin{vmatrix} -\sin\lambda_1 l_1 & 0 & 0 \\ \lambda_1 & 0 & -\lambda_2 \\ 0 & \cos\lambda_2 l_2 & \sin\lambda_2 l_2 \end{vmatrix}=0$$

or

$$\lambda_1\cos\lambda_1 l_1\sin\lambda_2 l_2+\lambda_2\sin\lambda_1 l_1\cos\lambda_2 l_2=0$$

where

$$\lambda_2=\sqrt{\frac{P}{EI_2}}$$

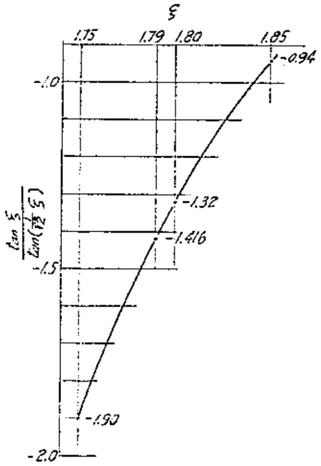


Fig. 8

or

$$P=\frac{2\times(1.79)^2}{\pi^2}\cdot\frac{\pi^2EI_2}{L^2}$$

$$=0.649\frac{\pi^2EI_2}{L^2}\text{ .... }(34)$$

Now, we may know that the plate shall have about 65% strength of the rectangular plate with I_2.

PART II. MISCELLANEOUS APPLICATIONS.

1. Column with Linearly Varying Moment of Inertia.

Let us consider a column with curved flanges as shown in **Fig. 1**, and assume that

$$I=\frac{I_0}{l_0}x=2F_0z^2$$

$$I_0=2F_0z_0^2$$

or

$$z=z_0\sqrt{\frac{x}{l_0}}$$

where I=moment of inertia at x,

F_0=one flange area.

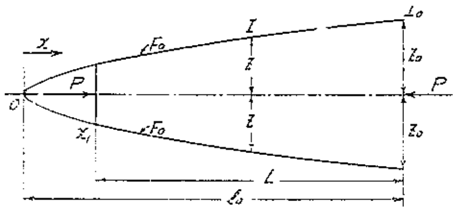


Fig. 1

Then, the stability condition for the truncated triangular plate (see, art. 3, Part I) will be applied to the present case, and we may write the following equations.

For $0< x_1 <l_0$,

$$J_1(\sqrt{4\lambda x_1})\,N_1(\sqrt{4\lambda l_0})-J_1(\sqrt{4\lambda l_0})\,N_1(\sqrt{4\lambda x_1})=0\text{ .... }(1)$$

and if $x_1=0$,

$$J_1(\sqrt{4\lambda l_0})=0\text{ .... }(2)$$

where x_1 and l_0 denote distances of both ends of the column from the origin.

On the latter case, we have already discussed in art. 3 (a), Part I, so we will now consider the former case only.

Rewritting eq. (1) as

$$J_1(\kappa\zeta)N_1(\kappa\zeta)-J_1(\kappa\zeta)N_1(\zeta)=0 \qquad (3)$$

where $\zeta=\sqrt{4\lambda x_1}$,

$$\kappa=\sqrt{\frac{l_0}{x_1}},$$

and let ζ_0 be the least root of this equation for a certain κ, then we shall be able to find the primary critical load P as follows.

Since,

$$\lambda=\frac{P}{EI_0}l_0,\quad x_1=\frac{l_0}{\kappa^2},\quad x_1=l_0-L,$$

and

$$l_0=\frac{\kappa^2}{\kappa^2-1}L,\qquad x_1l_0=\frac{\kappa^2}{(\kappa^2-1)^2}L^2,$$

$$P=\frac{\zeta_0^2(\kappa^2-1)^2}{4\kappa^2\pi^2}\cdot\frac{\pi^2EI_0}{L^2} \qquad (4)$$

where $\kappa^2=\dfrac{l_0}{x_1}$, and $0<x_1<l_0$,

L = effective length of the column.

As for the roots ζ_0, Janke-Emde-Funktionentafeln is to be conveniently referred to, or otherwise by graphical representation of the functions they will be obtained, and we have the following results.

$\kappa^2=\dfrac{l_0}{x_1}=$	∞	20	4	2.25	1.44	1.00
$\xi=\dfrac{1}{\kappa^2}=\dfrac{x_1}{l_0}=$	0	0.05	0.25	0.444	0.694	1.00
$\zeta_0=$	–	0.967	3·20	6.32	15.73	–
$\mu=$	0.372[1]	4.428	0.584	0.700	0.841	1.00

where $P=\mu\dfrac{\pi^2EI_0}{L^2}$.

The curve of μ will be traced in **Fig. 2.**

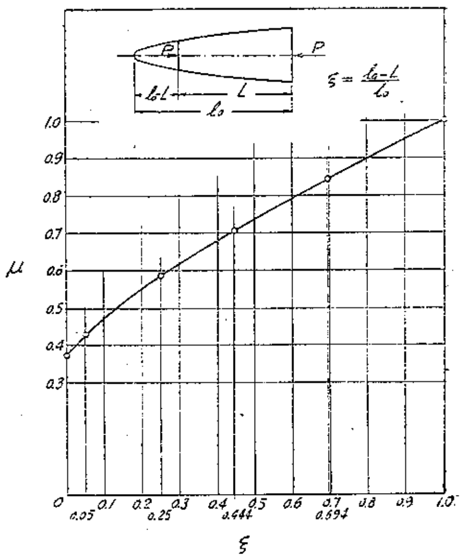


Fig. 2

(1) Part I, art. 3 (a) is to be referred to.

2. Column with Truncated Tapered Flanges.

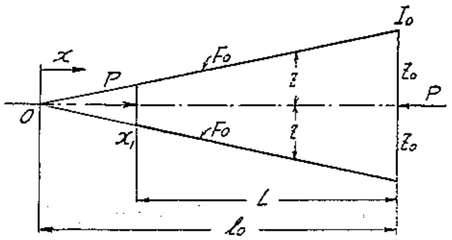


Fig. 3

In the present case, as shown in Fig. 3,

$$I=\frac{I_0}{l_0^2}x^2=2F_0z^2$$

$$I_0=2F_0z_0^2$$

and $z=\frac{z_0}{l_0}x$

And the differential equation of the virtual deflection of the column will be given as

$$\frac{EI_0}{l_0^2}x^2\frac{dy^2}{dx^2}+Py=0 \qquad (5)$$

or putting $\frac{Pl_0^2}{EI_0}=\alpha$

$$x^2\frac{dy^2}{dx^2}+\alpha y=0 \qquad (6)$$

The solution of this homogeneous equation is known as

$$y=\sqrt{x}\,\{A\cos(\beta\,lg\,x)+B\sin(\beta\,lg\,x)\}^{*} \qquad (7)$$

where $\beta=\frac{1}{2}\sqrt{4\alpha-1}$.

Hence, the conditional equation of stability for the truncated column, will be given as

$$\begin{vmatrix}\cos(\beta\,lg\,x_1) & \sin(\beta\,lg\,x_1)\\ \cos(\beta\,lg\,l_0) & \sin(\beta\,lg\,l_0)\end{vmatrix}=0 \qquad (8)$$

or

$$\sin\left(\beta\,lg\frac{l_0}{x_1}\right)=0 \qquad (9)$$

And we may know that the least root of this equation should be

$$\beta\,lg\,\frac{l_0}{x_1}=\pi \qquad (10)$$

Now, reminding that

$$\beta=\frac{\sqrt{4\alpha-1}}{2} \text{ and } \alpha=\frac{Pl_0^2}{EI_0}$$

we get

$$P=\left\{\frac{4}{\left(lg\frac{l_0}{x_1}\right)^2}+\frac{1}{\pi^2}\right\}\frac{(l_0-x_1)^2}{4l_0^2}\cdot\frac{\pi^2EI_0}{L^2} \qquad (11)$$

* lg denotes Napierian logarithm.

$$\text{or}\quad P=\left\{\frac{4}{\left(lg\frac{1}{\xi}\right)^2}+\frac{1}{\pi^2}\right\}\frac{(1-\xi)^2}{4}\cdot\frac{\pi^2EI_0}{L^2}\quad\ldots\ldots\ldots\ldots(12)$$

where $L=$ effective length of the column,

$=(l_0-x_1)$

and $\xi=\frac{x_1}{l_0}=\frac{l_0-L}{l_0}$.

P's for various truncated columns of this kind will be given as follows.

ξ	0	0.01	0.02	0.05	0.1	0.15
μ	0.025	0.071	0.088	0.123	0.173	0.219
ξ	0.20	0.40	0.60	0.80	1.00	
μ	0.263	0.438	0.617	0.801	1.000	

where $P=\mu\frac{\pi^2EI_0}{L^2}$

These values of μ are plotted in **Fig. 4**, in which we shall see the considerable difference between Bleich's μ and ours, especially when ξ is less than about 0.3.

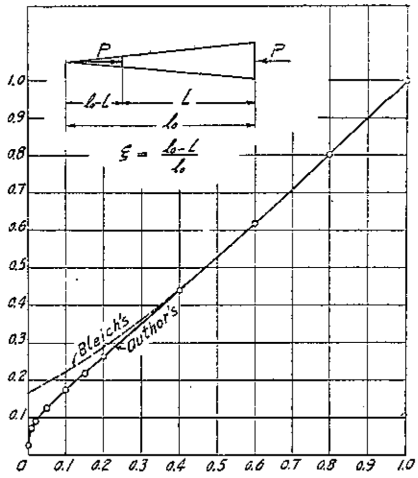


Fig. 4

3. Column with Truncated Rhombic Flanges.

In the present case, if we assume that the flanges of the column is symmetrically truncated as shown in **Fig. 5**, we may apply the condition of equilibrium given by eq. (16) in Part I, and by the previous discussions, we know that,

for $x_1 \leqq x \leqq l_0$,

$$y=\sqrt{x}\,\{A\cos(\beta\, lg\, x)+B\sin(\beta\, lg\, x)\}$$

and therefore

$$y'=\frac{1}{2\sqrt{x}}[A\{\cos(\beta\, lg\, x)-2\beta\sin(\beta\, lg\, x)\}$$
$$+\mathbf{B}\{\sin(\beta\, lg\, x)+2\beta\cos(\beta\, lg\, x)\}]\ldots(13)$$

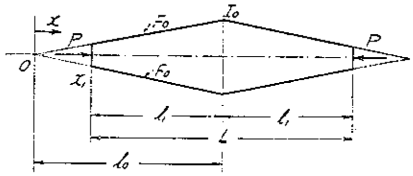


Fig. 5

Consequently, by condition of

$$y=0 \qquad \text{at } x=x_1$$
$$\text{and}\quad y'=0 \qquad \text{at } x=l_0$$

we have

$$\begin{vmatrix} \cos(\beta\, lg\, x_1) & \sin(\beta\, lg\, x_1) \\ \cos(\beta\, lg\, l_0) - 2\beta\sin(\beta\, lg\, l_0) & \sin(\beta\, lg\, l_0) + 2\beta\cos(\beta\, lg\, l_0) \end{vmatrix} = 0 \qquad \ldots\ldots(14)$$

or simply we may rewrite as

$$\sin\left(\beta\, lg\frac{1}{\xi} + \gamma\right) = 0 \qquad \ldots\ldots(15)$$

where $\gamma = \tan^{-1} 2\beta$,

$$\frac{1}{\xi} = \frac{l_0}{x_1}.$$

Therefore, the primary critical load will be given by solving the next equation with respect to β.

$$\beta\, lg\frac{1}{\xi} + \tan^{-1} 2\beta = \pi \qquad \ldots\ldots(16)$$

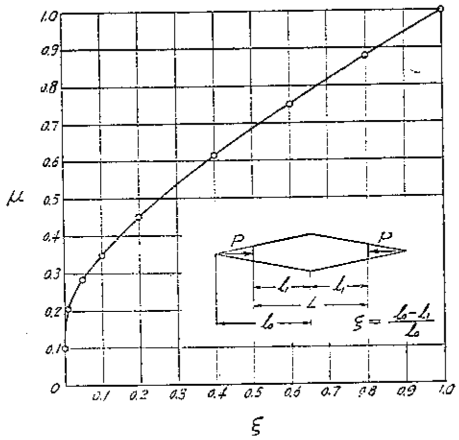


Fig. 6

To solve this equation, let us expand it by well known Lagrange's theorem, then we get

$$\lambda = b - a\tan^{-1} b + \frac{a^2}{1+b^2}\tan^{-1} b - \ldots \qquad \ldots\ldots(17)$$

where

$$a = \frac{2\pi}{lg\frac{1}{\xi}}, \quad b = \frac{2}{lg\frac{1}{\xi}},$$

$$\lambda = 2\beta = \sqrt{4\alpha - 1}, \quad \alpha = \frac{Pl_0^2}{EI_0}.$$

By this series and eq. (16), we can get easily the following results for different values of ξ.

$\xi=0$	0.01	0.05	0.1	0.2	0.4	0.6	0.8	1.0
$\lambda=0$	1.02	1.45	1.80	2.44	3.98	6.73	14.7	—
$\mu=0.101$	0.203	0.284	0.348	0.451	0.614	0.750	0.880	1.00

where $P = \mu\frac{\pi^2 EI_0}{L^2}$,

and $\mu = \frac{\lambda^2+1}{\pi^2}\left(\frac{l_1}{l_0}\right)^2 = \frac{\lambda^2+1}{\pi^2}(1-\xi)^2.$

The curve of μ will be traced in **Fig. 6.**

4. Column with Truncated Lenticular Flanges.

In the present case, we assume that the flanges trace the parabolic curves or that a column may have the following relations.

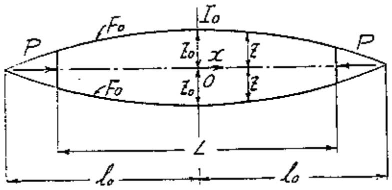


Fig. 7

$$I = I_0\left\{1-\left(\frac{x}{l_0}\right)^2\right\}^2 = 2F_0 z^2$$

$$I_0 = 2F_0 z_0^2$$

$$z = \sqrt{2F_0}\left\{1-\left(\frac{x}{l_0}\right)^2\right\} z_0$$

The differential equation of the virtua deflection is now written as

$$EI_0\left\{1-\left(\frac{x}{l_0}\right)^2\right\}^2 \frac{dy^2}{dx^2} + Py = 0 \quad \cdots\cdots (18)$$

or putting $\xi = \frac{x}{l_0}$ and $\alpha = \frac{Pl_0^2}{EI_0}$, we get

$$(1-\xi^2)^2 \frac{dy^2}{d\xi^2} + \alpha y = 0 \quad \cdots\cdots (19)$$

Equation (19) is the well known differential equation which has been completely discussed by Dr. Zimmermann(1) in his vibration theory, and we know that so for as $\alpha > 1$

$$y = \sqrt{1-\xi^2}\,\{A\cos\kappa w + B\sin\kappa w\} \quad \cdots\cdots (20)$$

where $\kappa = \sqrt{\alpha - 1}$

and $w = \text{arc}\tanh\xi$.

Consequently, for the column hinged at any two points x_1 and x_2, the condition of the stability will be given as

$$\begin{vmatrix} \cos\kappa w_1 & \sin\kappa w_1 \\ \cos\kappa w_2 & \sin\kappa w_2 \end{vmatrix} = 0 \quad \cdots\cdots (21)$$

or

$$\sin\kappa(w_2 - w_1) = 0 \quad \cdots\cdots (22)$$

where $w_1 = \text{arc}\tanh\xi_1, \quad \xi_1 = \frac{x_1}{l_0}$

$w_2 = \text{arc}\tanh\xi_2, \quad \xi_2 = \frac{x_2}{l_0}$

and we know that the least critical load shall be given by

$$\kappa(w_2 - w_1) = \pi.$$

(1) Dr. Zimmermann—Schwingungen eines Trägers, 1896.

Now, reminding that

$$\kappa=\sqrt{\alpha-1}$$

we have

$$\alpha=\left\{\frac{\pi}{\text{arc tanh }\bar{\xi}_2-\text{arc tanh }\bar{\xi}_1}\right\}^2+1 \quad\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

or

$$P=\left\{\frac{1}{(\text{arc tanh}\bar{\xi}_2-\text{arc tanh}\bar{\xi}_1)^2}+\frac{1}{\pi^2}\right\}(\bar{\xi}_2-\bar{\xi}_1)^2\frac{\pi^2EI_0}{L^2}. \quad\ldots\ldots\ldots\ldots(24)$$

This is the general expression of the primary critical load of the truncated lenticular column with parabolic flanges and hinges at both ends. We will discuss further in the following different two cases.

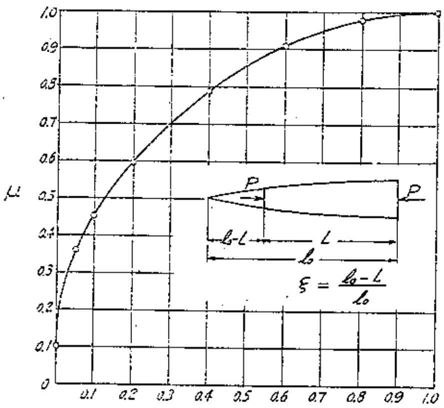


Fig. 8

(a) Column with Hinges at $x=0$ and $x=L$, as Shown in Fig. 8.

In this case, putting

$$\bar{\xi}_1=0 \quad\text{and}\quad \bar{\xi}_2=\frac{L}{l_0},$$

we get by eq. (24),

$$P=\left\{\frac{1}{\left(\text{arc tanh}\frac{L}{l_0}\right)^2}+\frac{1}{\pi^2}\right\}\left(\frac{L}{l_0}\right)^2\frac{\pi^2EI_0}{L^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

By this equation we get the following numerical values for various columns of this kind.

$\frac{L}{l_0}$	1	0.95	0.9	0.8	0.6	0.4	0.2	0
$\xi=\frac{l_0-L}{l_0}$	0	0.05	0.1	0.2	0.4	0.6	0.8	1.0
μ	0.101	0.360	0.456	0.595	0.786	0.906	0.975	1.00

where $P=\mu\frac{\pi^2EI_0}{L^2}$.

The curve of μ will be traced in **Fig. 8.**

(b) Symmetrically Truncated Lenticular Column.

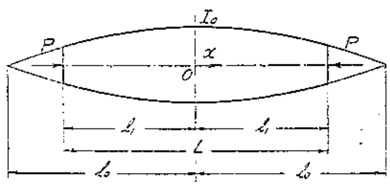


Fig. 9

In this case, putting

$$\bar{\xi}_1=+\frac{l_1}{l_0}$$

$$\bar{\xi}_2=-\frac{l_1}{l_0}$$

and $\quad 2l_1=L$

in eq. (24), we have the following expression for the primary critical load.

$$P=\left\{\frac{1}{4\left(\operatorname{arc\,tanh}\frac{l_1}{l_0}\right)^2}+\frac{1}{\pi^2}\right\}\left(\frac{L}{l_0}\right)^2\frac{\pi^2EI_0}{L^2} \qquad (26)$$

By this equation we get the following numerical values.

$\frac{l_1}{l_0}$	1.00	0.95	0.9	0.8	0.6	0.4	0.2	0
$\xi=\frac{l_0-l_1}{l_0}$	0	0.05	0.1	0.2	0.4	0.6	0.8	1.00
μ	0.405	0.635	0.702	0.789	0.896	0.955	0.987	1.00

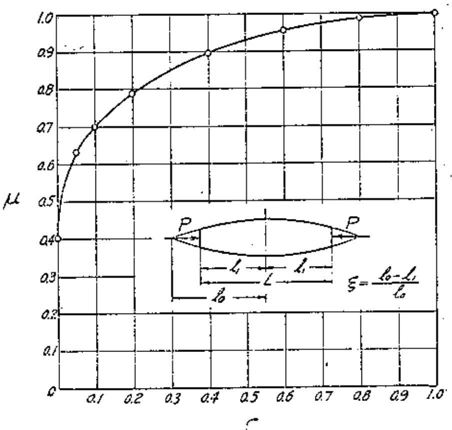


Fig. 10

where $P=\mu\frac{\pi^2EI_0}{L^2}$.

The curve of μ will be traced in **Fig. 10.**

PART III. EXPERIMENTS.

1. Preface.

The present part of this paper covers the report of our experiments and their results, which have been performed by the author in the Research Office of the Japanese Government Railways, in order to support our theoretical results obtained in Parts I and II.

2. Test Pieces and Experiment Devices.

(a) Test Pieces.

We have prepared a number of test pieces, cut from the galvanised sheet iron about 0.65 mm. thick, and their shapes and dimensions are as shown in **Figs. 1, 2** and **3** on **Pl. I.**

(b) Experiment Devices.

We have built up two different devices, one for testing in vertical position, and the other for testing in horizontal position, and their sketches are as shown in **Fig. 1** and **Fig. 2.** on **Pl. II.**

3. Results of Experiments.

Test Pieces	ξ	L Length (mm.)	$\frac{1}{2}P_c$ = $\frac{1}{2}$(critical load) (gr.)	$\mu=\dfrac{P_c\left(\frac{L}{500}\right)^2}{P_c \text{ of } S_0}$	Theoretical μ	Experiments for Part	Article
S_0	1	500	757.5	——	——		
S_1	—	500	491	0.648	0.649	I	3 (*d*)
A_0	0	500	274	0.362	0.372	I.	3 (*a*) (*b*)
A_1	0.05	500	330	0·436	0.428	II.	1
A_2	0.25	500	450	0.594	0.584		
B_0	0	500	443	0.585	0.584	I.	3 (*c*)
C_2	0.05	475	109	0.130	0.123		
C_3	0.10	450	161	0.172	0.173	II.	2
C_4	0.15	425	229	0.218	0.219		
D_1	0.025	487.5	191	0.240	——		
D_2	0.05	475	229	0.273	0.284	II.	3
D_3	0.1	450	330	0.353	0.349		
E_1	0.025	487.5	240	0.301	——		
E_2	0.05	475	291	0.347	0.360	II.	4 (*a*)
E_3	0.10	450	401	0.429	0.456		
E_4	0.20	400	713	0.602	0.595		
G_1	0.05	475	$645\times\frac{1}{1.28}$*	0.600	0.635		
G_2	0.10	450	$825\times$ 〃	0.688	0.702	II.	4 (*b*)
G_3	0.15	425	1 024 $\times$ 〃	0.763	——		
G_4	0.20	400	1 219 $\times$ 〃	0.805	0.789		

PART IV. CONCLUSIONAL REMARKS.

1. Summaries.

In Part I, we have discussed the fundamental principles on the stability conditions of the column and shown some examples of their applications to the plates with variable sections.

In Part II, we have performed the further miscellaneous applications to the columns with tapered or curved flanges, and also the calculations of μ by which we shall be able to know the strength of such columns as

$$P=\mu\frac{\pi^2 EI_0}{L^2},$$

where

* Correction for difference of thickness of test pieces $=\left(\dfrac{\text{thickness of } S_0}{\text{thickness of test piece}}\right)^3$.

$\frac{\pi^2 EI_0}{L^2}$ = Euler's critical load for the column with a constant moment of inertia I_0.

In Part III, referring to the results of our experiments, we have shown the validity of our theoretical investigations.

2. Notes on Practical Applications.

(**a**) In the present paper, we have discussed only for the cases in which the fundamental differential equations are mathematically soluble, while so far as we believe, they might greatly help to check some of the results of the anxious numerical calculations even when they are solved by a method of numerical integrations.

(**b**) When we have to consider the plastic deformation or plastic strength of a column, the elastic modulus E should be replaced by the Engesser's "Knick modul T".(1)

(**c**) In the structural engineering, the useful applications of the present theories might be specially expected in the designs of the high viaduct piers, as in a cantilever bridge etc, and of towers of the suspension bridges etc. In our country where the effect of the earthquake is dangerous, the preference of the steel piers to the high concrete piers is quite evident, and we feel that the present theories might have more cases of the applications in the future.

APPENDIX TO PART II.

1. Columns with Concave Flanges.

We shall now consider the columns as shown in **Fig. 1**, where

$$I = 2F_0 z^2, \qquad I_0 = 2F_0 z_0^2,$$

and

$$z = \frac{z_0}{e^{\frac{x}{l_0}}} = \frac{z_0}{e^{\xi}}, \qquad \xi = \frac{x}{l_0},$$

(1) Engesser-Über die Knickfestigkeit gerader Stäbe, Z. Arch. Ing. Wes. 1889.

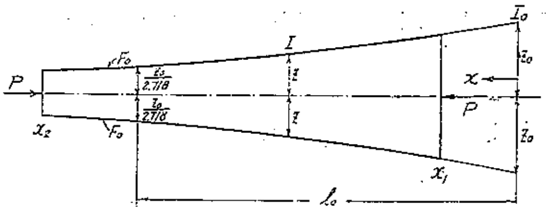


Fig. 1

$l_0=\left(\text{such length as } z=\frac{z_0}{e} \text{ at } x=l_0\right)$,

$e=$ base of the Napierian logarithm or 2.718.

In the present case, the fundamental differential equation, $EI(x)\frac{dy^2}{dx^2}+Py=0$, will be written as

$$EI_0 e^{-2\xi}\frac{dy^2}{dx^3}+Py=0$$

or

$$\frac{EI_0}{l_0^2}e^{-2\xi}\frac{dy^2}{d\xi^2}+Py=0 \quad \text{.............................(1)}$$

Now putting $\frac{Pl_0^2}{EI_0}=\alpha$, we have

$$e^{-2\xi}\frac{dy^2}{d\xi^2}+\alpha y=0 \quad \text{....................................(2)}$$

To find the primitive of this equation, put

$$\sqrt{\alpha}\,.\,e^{\xi}=u.$$

Then, we get from eq. (2),

$$\frac{dy^2}{du^2}+\frac{1}{u}\,\frac{dy}{du}+y=0 \quad \text{..............................(3)}$$

and we may know that

$$y=AJ_0(\sqrt{\alpha}.e^{\xi})+BY_0(\sqrt{\alpha}.e^{\xi}) \quad \text{........................(4)}$$

Therefore, for the columns with two hinges at x_1 and x_2, the condition of the critical equilibrium will be given as

$$J_0(\sqrt{\alpha}.e^{\xi_1})Y_0(\sqrt{\alpha}.e^{\xi_2})-J_0(\sqrt{\alpha}.e^{\xi_2})Y_0(\sqrt{\alpha}.e^{\xi_1})=0 \quad \text{........(5)}$$

where $\xi_1=\frac{x_1}{l_0}$ and $\xi_2=\frac{x_2}{l_0}$.

And similarly for the columns fixed at x_1 and hinged at x_2, we get

$$J_0'(\sqrt{\alpha}.e^{\xi_1})Y_0(\sqrt{\alpha}.e^{\xi_2})-J_0(\sqrt{\alpha}.e^{\xi_2})Y_0'(\sqrt{\alpha}.e^{\xi_1})=0 \quad \text{........(6)}$$

Numerical example.

To find the primary critical load of the column with hinges at $x=0$ and $x=l_0$ as shown in **Fig. 2.**

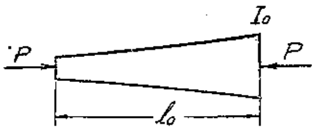


Fig. 2

In the present case, putting $\xi_1=0$ and $\xi_2=1$ in eq. (5), we have

$$J_0(\sqrt{\alpha})Y_0(\sqrt{\alpha}.e)-J_0(\sqrt{\alpha}.e)Y_0(\sqrt{\alpha})=0 \quad ..\ (7)$$

Now if we put $\alpha=3.25$, we have $\sqrt{\alpha}=1.80$, $\sqrt{\alpha}.e=4.89$ and

$$J_0(\sqrt{\alpha})Y_0(\sqrt{\alpha}.e)=-0.168,$$

$$J_0(\sqrt{\alpha}.e)Y_0(\sqrt{\alpha})=-0.167,$$

whence we know that the least root of eq. (7) will be

$$\alpha=3.25$$

and that the primary critical load will be given as follows.

$$\frac{Pl_0^2}{EI_0}=3.25$$

or

$$P=0.329\frac{\pi^2 EI_0}{l_0^2}$$

2. On Higher Critical Loads of the Columns with Variable Cross Sections.

In the previous articles, we have made various considerations on the primary critical strength of the columns with variable cross sections, while the higher critical loads of such columns will be similarly obtained by taking the greater roots of the conditional equations of the critical equilibrium, as shown in the following examples.

1. Column with Linearly Varying Moment of Inertia.

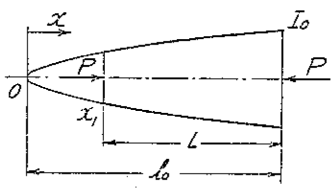


Fig. 3

By eq. (4) on page 9.

$$P=\frac{\zeta_0(\kappa^2-1)^2}{4\kappa^2\pi^2}\cdot\frac{\pi^2 EI_0}{L^2}$$

where $\kappa^2=\dfrac{l_0}{x_1}$,

L=effective length and ζ_0= root of the following equation.

$$J_1(\zeta)N_1(\kappa\zeta)-J_1(\kappa\zeta)N_1(\zeta)=0,$$

where $\zeta=\sqrt{4\lambda x_1}$

and $\lambda=\dfrac{P}{EI_0}l_0$

and when $x_1=0$, by eq. (21) on page 5, we know that ζ_0 shall be root of $J_1(\sqrt{4\lambda L})=0$.

When $x_1=0$.

The roots of $J_1(\sqrt{4\lambda L})=0$ will be given as follows.

$\zeta_0=\sqrt{4\lambda L}=3.83,\quad 7.02,\quad 10.17,\quad 13.34,\ldots\ldots$* and we have

$$P=\frac{\zeta_0^2}{4\pi^2}\cdot\frac{\pi^2 EI_0}{L^2}$$

or

$$P=\mu\frac{\pi^2 EI_0}{L^2}$$

where

$$\mu=\frac{\zeta_0^2}{4\pi^2}$$

and

$$\mu=0.372,\quad 1.248,\quad 2.619,\quad 4.508,\ldots\ldots\ldots$$

When $x_1>0$

The roots of $J_1(\zeta)N_1)\kappa\zeta)-J_1(\kappa\zeta)N_1(\zeta)=0$ will be given as follows.

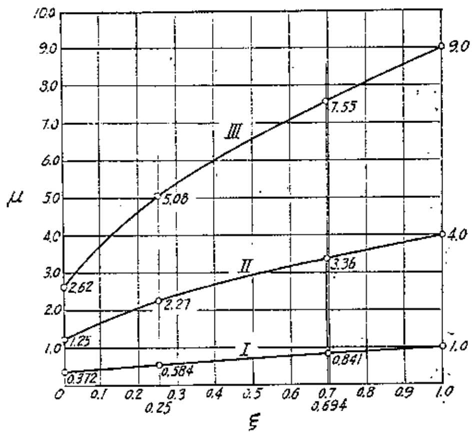
For $\kappa=2$,

$\zeta_0=3.20,\quad 6.31,\quad 9.44,\quad 12.58,\ldots\ldots\ldots\ldots$*

and for $\kappa=1.2$

$\zeta_0=15.73,\quad 31.43,\quad 47.13\quad 62.83,\ldots\ldots\ldots\ldots$*

Therefore, by



$$P=\mu\frac{\pi^2 EI_0}{L^2}$$

where

$$\mu=\frac{\zeta_0(\kappa^2-1)^2}{4\kappa^2\pi^2},$$ we have

for $\kappa=2$,

$\mu=0.584,\quad 2.269,\quad 5.079,\ldots\ldots$

and for $\kappa=1.2$,

$\mu=0.841,\quad 3.356,\quad 7.547,\ldots\ldots$

The values of μ for the higher critical loads will be traced in the accompanying diagram.

* See Janke-Emde-Funktionen tafeln.

2. Column with Tapered Flanges.

By eq. (9) on page 10, the higher critical loads will be given by

$$\beta\, lg\frac{l_0}{x_1}=n\pi$$

or after the similar reduction, we have

$$P=\left\{\frac{4n^2}{\left(lg\dfrac{l_0}{x_1}\right)^2}+\frac{1}{\pi^2}\right\}\frac{(l_0-x_1)^2}{4l_0^2}\frac{\pi^2EI_0}{L^2}$$

where $n=1,\ 2,\ 3,\ldots\ldots\ldots\ldots\ldots\ldots$

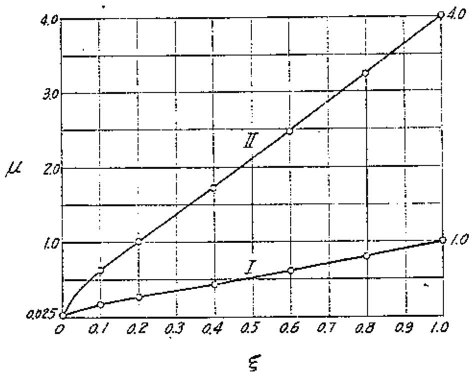
Now, writing $P=\mu\dfrac{\pi^2EI_0}{L^2}$

where

$$\mu=\left\{\frac{4n^2}{\left(lg\dfrac{l_0}{x_1}\right)^2}+\frac{1}{\pi^2}\right\}\frac{(l_0-x_1)^2}{4l_0^2},\ \text{we have}$$

	$\frac{x_1}{l_0}=\xi=0$	0.1	0.2	0.4	0.6	0.8	1.0
for $n=1$	$\mu=0.025$	0.173	0.263	0.438	0.617	0.801	1.00
and for $n=2$	$\mu=0.025$	0.631	1.005	1.717	2.457	3.203	4.00

The values of μ will be traced in the following diagram.



3. Column with Symmetrically Truncated Lenticular Flanges.

By eq. (22) on page 13, we may put for the higher critical loads as follows.

$$\kappa(w_2-w_1)=n\pi$$

and in the present case, after the similar reduction to page 15, we have

$$P=\left\{\frac{n^2}{4\left(\text{arc tanh}\dfrac{l_1}{l_2}\right)^2}+\frac{1}{\pi^2}\right\}\left(\frac{L}{l_0}\right)^2\frac{\pi^2EI_0}{L^2}$$

or

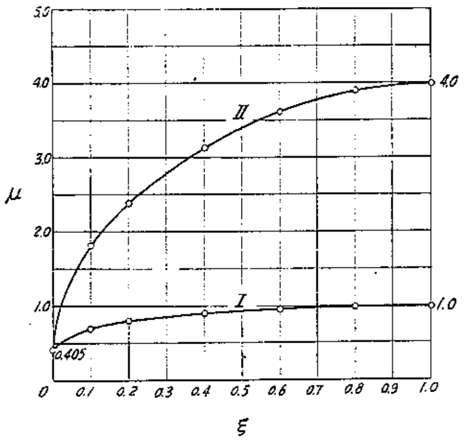
$$P=\mu\frac{\pi^2EI_0}{L^2},$$

where

$$\mu=\left\{\frac{n^2}{4\left(\text{arc tanh}\frac{l_1}{l_0}\right)^2}+\frac{1}{\pi^2}\right\}\left(\frac{L}{l_0}\right)^2.$$

$\frac{l_0-l_1}{l_0}=\xi=0$		0.1	0.2	0.4	0.6	0.8	1.0
for $n=1$	$\mu=0.405$	0.702	0.789	0.896	0.955	0.987	1.00
for $n=2$	$\mu=0.405$	1.825	2.379	3.144	3.624	3.900	4.00

The values of μ will be shown in the next diagram.



Conclusional Remarks.

By these three typical examples, we shall know that the higher critical loads P_n of such columns as shown above shall have the following relation.

$$P_n<n^2\mu,\frac{\pi^2EI_0}{L^2}$$

or $$P_n<n^2P_1$$

where P_1=primary critical load, while P_n tends to n^2P_1 as ξ approaches to unity.

Pl. I

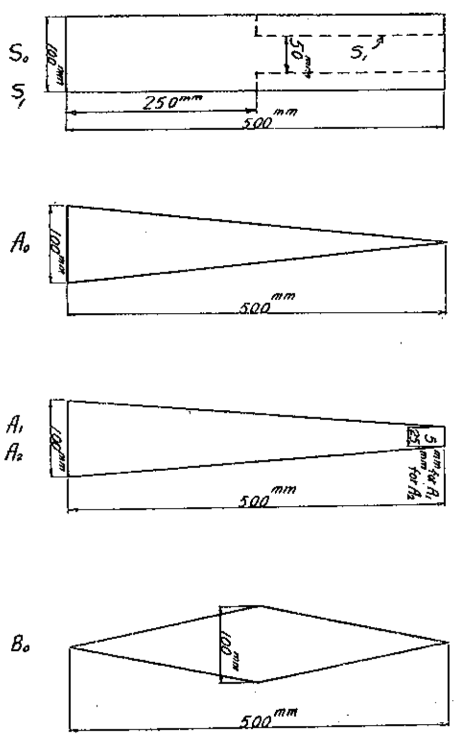


Fig. 1

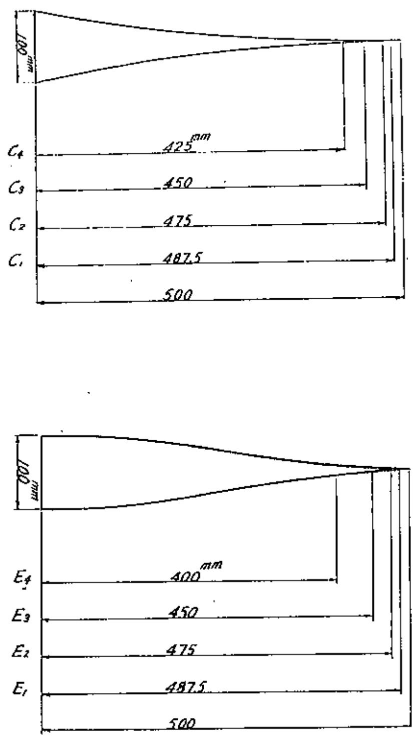


Fig. 2

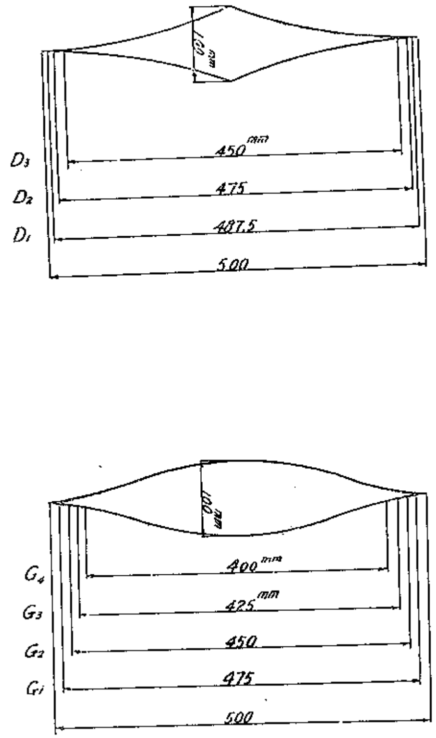
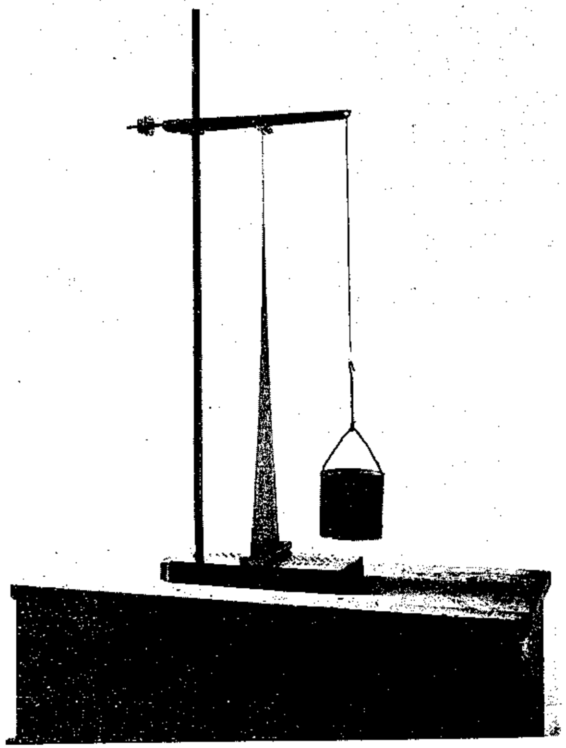
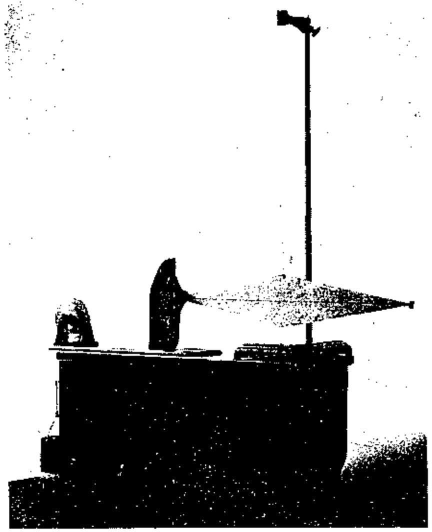


Fig. 3

(土木學會誌第十五卷第三號附圖)

Pl. II





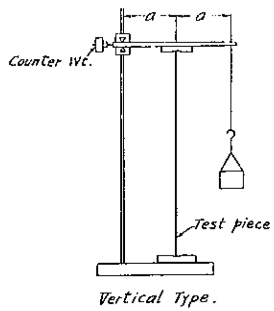


Fig. 1

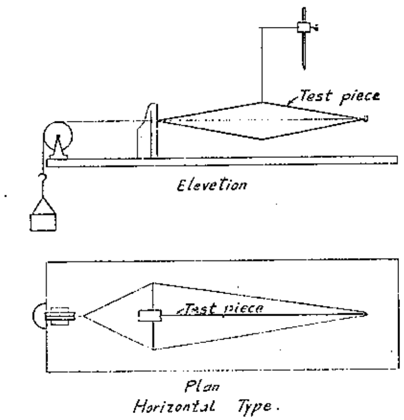


Fig. 2