論 說 報 告

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STRESSES IN SUBAQUEOUS TUNNELS BUILT IN THE WATER-BEARING SOIL

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Synopsis

The writer being engaged in the investigation and designing of a subaqueous tunnel, to be built under the Strait of Shimonoseki, had to check the comparative strength of the sections designed for different methods of construction.

However, not being able to find any published rational formulas to guide the calculation, he had at last to evolve a new one, in order to assure the safety of the section to be adopted, not only for the ordinary loading of dead load, water pressure and superincumbent earth, but also for the live load and the superimposed load, which might come to bear on the tunnel from sunken vessels in the said strait.

This paper, containing these formulas, is presented to the engineering society that they may be fully discussed and used advantageously in designing similar structures which may be built in the future.

Contents

INTRODUCTION

Since the first subaqueous tunnel under Thames River was built by Isambard Brunnel in 1825–1843, great progress has been made in the art of tunneling under the water, and various methods were also evolved by many engineers to make the work safe and easy and more economical; and the construction of subaqueous tunnel is not now an adventure as it was thought before, but an engineering work that can be achieved successfully with confidence, though in some case many difficulties have to be overcome according to the locality.

Nevertheless, there is only little written on the line of rational treatment of the stresses that will be produced in this kind of structure, and the proportioning of the section is usually made after the preceeding examples in the similar locality and judgement of engineers gained from the experiences in the actual construction.

This is principally due to the uncertainty of the external forces to be assumed to act on the tunnel lining, because the earth overlying the tunnel section may support itself in a certain degree by cohesion and not exert its whole weight as the external loading, and the amount of lateral pressure from the surrounding soil is also a matter of arbitrary assumption.

However, by assuming the surrounding soil to be entirely cohesionless particles permeated with water and treating earth and water separately, such a uncertainty will be removed; and though it will give, in some case, an error on safety side, it will be proper to assume so, because the depth of earth over the tunnel is usually rather shallow to depend on its arching action and the presence of water will make the cohesion of earth entirely unreliable.

Under the consideration explained above, it will be assumed in this paper that the water pressure to act on all sides of the section in the normal direction, and the earth with its weight reduced from buoyancy is assumed to act vertically on the tunnel section.

As to the reaction, which will keep the tunnel in equilibrium with all vertical loadings, it is assumed to act uniformly and normal to its bottom surface, and its intensity is determined that sum of the vertical components of all reactions is equal to the total vertical loading.*

This is the fundamental assumption about external forces, and the object of this paper is to deduce necessary formulas, under this assumption, to aid for rational analysis of the tunnel stresses.

For the sake of convenience, the paper is divided into the following three sections.

1. General equations.
2. Application of general equations to the circular tube with uniform cross-section.
3. Application of general equations to the rectangular tunnels with circular bore.

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* By bottom surface is meant the surface below the points where vertical plane is tangent at either side of the section, reaction to the vertical force being assumed to act only over this surface.

When bottom of the section is a horizontal plane, reaction is evidently of uniform intensity and in normal or vertical direction; when it is a curved surface, we can consider it to consist of polygonal surfaces of infinite number, and since the vertical force will press the section against surrounding soil by wedge action, its reaction will act normal to these surfaces in equal intensity, for in waterbearing soil, pressure is supposed to be transmitted by water, which will be, by Pascal's law, of equal intensity in all directions and normal to the surface on which it acts.

The best treatise that the writer knows on the subject is the paper of Professor Steiner on the circular tube, „Beitrag zur Theorie des Röhrentunnels kreisformigen Querschnittes," published 1906 in Prag. He assumed earth as cohesionless particles acting vertically, and the water pressure normal all way around as stated before. But he assumes reaction to the vertical force to be uniform horizontally over the projection of the tube, and not normal to the surface of lower half circle.

This assumption is not natural especially in water-bearing soil and will give too great stresses for the section. Moreover, Professor Steiner tried to find the effect of dead load, water pressure, earth load, and reaction separately, and in each case he had, for sake of equilibrium, to assume the bottom center point to be hinged in place, instead of assuming distributed reaction over the surface of the tube.

From the difference of these assumptions, formulas obtained by the writer are different from that given by the Professor, and the point of maximum moment in the tube is also different, and readers may do well compare this with that of Professor Steiner.

GENERAL EQUATIONS

1. Moments and Thrusts

Let *ABCD* in **Fig. 1.** represent the neutral axis of a section of a subaqueous tunnel, and P_1, P_2, P_3, etc. be the external forces acting on the section.

The crown point *A* will be subjected to the unknown moment, thrust and shear (M_0, H_0, V_0) from these external forces.

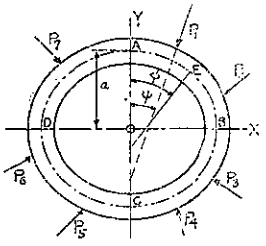


Fig. 1

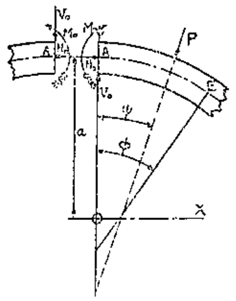


Fig. 2

Now, if we suppose the section is cut at *A* and the moment M_0, thrust H_0, and shear V_0 are applied at this point the same equilibrium as before will be maintained.

Let

OX, OY: horizontal and vertical axes,

x, y: co-ordinates of any point E on the neutral axis,

ϕ: angle, normal at E makes with vertical axis,

ψ: angle, force P makes with vertical axis,

m: sum of moments of all external forces acting between A and E; moment being taken positive for clockwise rotation or when causing compression on outside fiber, and negative for counter-clockwise rotation or when causing tension on outside fiber.

Then, the moment and thrust at E are

$$M=M_0+V_0x+H_0(a-y)+m \quad \text{.......................}(1)$$

$$N=H_0\cos\phi-V_0\sin\phi+\Sigma P\sin(\phi-\psi) \quad \text{...............}(2)$$

Let

A: cross-sectional area at E,

I: moment of inertia at E,

E: modulus of elasticity of the material,

then the work of internal stresses at the section E is

$$\frac{M^2ds}{2EI}+\frac{N^2ds}{2EA}$$

Total work of the entire section is

$$\omega=\int_0^{2\pi}\frac{M^2ds}{2EI}+\int_0^{2\pi}\frac{N^2ds}{2EA} \quad \cdots\cdots(3)$$

By the principle of least work of Castigliano,

$$\frac{\partial\omega}{\partial M_0}=0,\quad \frac{\partial\omega}{\partial V_0}=0,\quad \frac{\partial\omega}{\partial H_0}=0$$

Substituting the values of M and N given in the equations (1) and (2) in the equation (3), and by partial differentiation we get the following equations. (E, being constant, is droped out of the equations.)

$$\left.\begin{aligned}
&M_0\int_0^{2\pi}\frac{ds}{I}+V_0\int_0^{2\pi}\frac{xds}{I}+H_0\int_0^{2\pi}\frac{(a-y)ds}{I}+\int_0^{2\pi}\frac{mds}{I}=0\\
&M_0\int_0^{2\pi}\frac{xds}{I}+V_0\left\{\int_0^{2\pi}\frac{x^2ds}{I}+\int_0^{2\pi}\frac{\sin^2\phi ds}{A}\right\}+H_0\left\{\int_0^{2\pi}\frac{x(a-y)ds}{I}\right.\\
&\qquad\left.-\int_0^{2\pi}\frac{\sin\phi\cos\phi\, ds}{A}\right\}+\int_0^{2\pi}\frac{mxds}{I}-\int_0^{2\pi}\frac{P\sin\phi\sin(\phi-\psi)\,ds}{A}=0\\
&M_0\int_0^{2\pi}\frac{(a-y)ds}{I}+V_0\left\{\int_0^{2\pi}\frac{x(a-y)\,ds}{I}-\int_0^{2\pi}\frac{\sin\phi\cos\phi\, ds}{A}\right\}\\
&\qquad+H_0\left\{\int_0^{2\pi}\frac{(a-y)^2ds}{I}+\int_0^{2\pi}\frac{\cos^2\phi ds}{A}\right\}+\int_0^{2\pi}\frac{m(a-y)ds}{I}\\
&\qquad+\int_0^{2\pi}\frac{P\cos\phi\sin(\phi-\psi)\,ds}{A}=0
\end{aligned}\right\}\quad\cdots(4)$$

Hence, when the form of the neutral axis of a tunnel section is known and external forces are given, we can determine three unknown quantities M_0, V_0, and H_0 from the above three equations, and then moment and thrust at any other point on the neutral axis will be given from the equations (1) and (2).

Usually tunnel section and external forces are symmetrical about the vertical axis OY, and in that case, since V_0 is equal to zero and the integration need only be carried one half of the section, the equation (4) is simplified to the following form.

$$\left.\begin{aligned}&M_0\int_0^{\pi}\frac{ds}{I}+H_0\int_0^{\pi}\frac{(a-y)\,ds}{I}+\int_0^{\pi}\frac{m\,ds}{I}=0\\&M_0\int_0^{\pi}\frac{(a-y)\,ds}{I}+H_0\left\{\int_0^{\pi}\frac{(a-y)^2ds}{I}+\int_0^{\pi}\frac{\cos^2\phi\,ds}{A}\right\}\\&\qquad+\int_0^{\pi}\frac{m(a-y)\,ds}{I}+\int_0^{\pi}\frac{P\cos\phi\sin(\phi-\psi)\,ds}{A}=0\end{aligned}\right\}\quad\cdots(5)$$

M_0 and H_0 can be determined from equation (5), and the moment and thrust at any other point on the neutral axis are given by

$$\left.\begin{aligned}&M=M_0+H_0(a-y)+m\\&N=H_0\cos\phi+\Sigma P\sin(\phi-\psi)\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(6)$$

Equation (5) can also be derived by assuming right half of the section as an arch with its end constrained that the direction of tangent at the point A to remain unchanged, for $\dfrac{\partial\omega}{\partial H_0}$ is the displacement of the point A in the direction of OX axis and $\dfrac{\partial\omega}{\partial M_0}$ the angular displacement of the point A, and these values must be equal to zero.

When the tunnel section is not uniform along the neutral axis, divide half section along the axis into n equal parts of the length Δs, and we get from equation (5)

$$\left.\begin{aligned}&M_0\sum_1^n\frac{1}{I}+H_0\sum_1^n\frac{a-y}{I}+\sum_1^n\frac{m}{I}=0\\&M_0\sum_1^n\frac{a-y}{I}+H_0\sum_1^n\left\{\frac{(a-y)^2}{I}+\frac{\cos^2\phi}{A}\right\}\\&\qquad+\sum_1^n\left\{\frac{m(a-y)}{I}+\frac{P\cos\phi\sin(\phi-\psi)}{A}\right\}=0\end{aligned}\right\}\cdots\cdots\cdots(7)$$

From which we get, at once

$$\left.\begin{aligned}M_0&=\frac{\sum_1^n\frac{a-y}{I}\cdot\sum_1^n\left\{\frac{m(a-y)}{I}+\frac{P\cos\phi\sin(\phi-\psi)}{A}\right\}-\sum_1^n\frac{m}{I}\cdot\sum_1^n\left\{\frac{(a-y)^2}{I}+\frac{\cos^2\phi}{A}\right\}}{\sum_1^n\frac{1}{I}\cdot\sum_1^n\left\{\frac{(a-y)^2}{I}+\frac{\cos^2\phi}{A}\right\}-\left\{\sum_1^n\frac{a-y}{I}\right\}^2}\\H_0&=\frac{\sum_1^n\frac{a-y}{I}\cdot\sum_1^n\frac{m}{I}-\sum_1^n\frac{1}{I}\cdot\sum_1^n\left\{\frac{m(a-y)}{I}+\frac{P\cos\phi\sin(\phi-\psi)}{A}\right\}}{\sum_1^n\frac{1}{I}\cdot\sum_1^n\left\{\frac{(a-y)^2}{I}+\frac{\cos^2\varphi}{A}\right\}-\left\{\sum_1^n\frac{a-y}{I}\right\}^2}\end{aligned}\right\}\cdots(8)$$

Since the effect of axial stress is very small compared to the effect of bending moment, by neglecting terms containing the former from the equation (8), we get

$$\left.\begin{aligned} M_0 &= \frac{\sum_1^n \frac{a-y}{I} \cdot \sum_1^n \frac{m(a-y)}{I} - \sum_1^n \frac{m}{I} \cdot \sum_1^n \frac{(a-y)^2}{I}}{\sum_1^n \frac{1}{I} \cdot \sum_1^n \frac{(a-y)^2}{I} - \left\{\sum_1^n \frac{a-y}{I}\right\}^2} \\ H_0 &= \frac{\sum_1^n \frac{a-y}{I} \cdot \sum_1^n \frac{m}{I} - \sum_1^n \frac{1}{I} \cdot \sum_1^n \frac{m(a-y)}{I}}{\sum_1^n \frac{1}{I} \cdot \sum_1^n \frac{(a-y)^2}{I} - \left\{\sum_1^n \frac{a-y}{I}\right\}^2} \end{aligned}\right\} \cdots\cdots\cdots (9)$$

When the tunnel section has the uniform cross-section, its moment of inertia is also constant, and we get from equation (5),

$$\left.\begin{aligned} & M_0\int_0^\pi ds + H_0\int_0^\pi (a-y)\,ds + \int_0^\pi m\,ds = 0 \\ & M_0\int_0^\pi (a-y)ds + H_0\left\{\int_0^\pi (a-y)^2 ds + \frac{I}{A}\int_0^\pi \cos^2\phi\,ds\right\} \\ & \qquad + \int_0^\pi m(a-y)\,ds + P\frac{I}{A}\int_0^\pi \cos\phi\sin(\phi-\psi)\,ds = 0 \end{aligned}\right\} \quad (10)$$

Neglecting the effect of axial stress as before, we get

$$\left.\begin{aligned} & M_0\int_0^\pi ds + H_0\int_0^\pi (a-y)ds + \int_0^\pi m\,ds = 0 \\ & M_0\int_0^\pi (a-y)\,ds + H_0\int_0^\pi (a-y)^2 ds + \int_0^\pi m\,(a-y)\,ds = 0 \end{aligned}\right\} \quad (11)$$

From equation (11), we get

$$\left.\begin{aligned} M_0 &= \frac{\int_0^\pi (a-y)ds \cdot \int_0^\pi m\,(a-y)ds - \int_0^\pi m\,ds \cdot \int_0^\pi (a-y)^2 ds}{\int_0^\pi ds \cdot \int_0^\pi (a-y)^2 ds - \left\{\int_0^\pi (a-y)\,ds\right\}^2} \\ H_0 &= \frac{\int_0^\pi m\,ds \cdot \int_0^\pi (a-y)ds - \int_0^\pi ds \cdot \int_0^\pi m(a-y)ds}{\int_0^\pi ds \cdot \int_0^\pi (a-y)^2 ds - \left\{\int_0^\pi (a-y)\,ds\right\}^2} \end{aligned}\right\} \quad (12)$$

Moment and axial stress at any point on the neutral axis are given by equation (6).

2. Stresses in the Lining

When the moment and thrust at any point on the neutral axis are known, the stresses in the lining can be determined from the equation

$$f=\frac{N}{A}\pm\frac{My}{I} \quad \dots\dots(13)$$

where

f: intensity of stresses at the point distant y from the neutral axis,

N: axial stress, M: bending moment,

A: cross-sectional area, I: moment of inertia.

Positive moment causes compression and negative moment causes tension on the outside fiber.

When the lining consist of uniform material, like concrete, the value of $y=\frac{1}{2}$ thickness will give the maximum fiber stress which should be within the limit allowable for the material.

If $\frac{My}{I}>\frac{N}{A}$, tensile stress will obtain on one side of the section, and some times it becomes necessary to use reinforcing steel to resist the same.

However, when this tensile stress is not very great, we can entirely neglect tensile resistance of concrete in the calculation, and in this case the resulting maximum compression will be determined by the following formula.

$$f=\frac{2N}{3\left(\frac{\text{thickness}}{2}-\frac{M}{N}\right)} \quad \dots\dots(14)$$

When tunnel section is made of composite structure, like reinforced concrete tube or cast iron segments with concrete lining, stresses in two constituent materials will be found as follows:

Let

E_c, A_c, I_c: modulus of elasticity, cross-sectional area and moment of inertia of concrete,

E_s, A_s, I_s: ditto of steel or iron,

E, A, I: ditto of composite structure,

M_c: moment to be carried by concrete,

M_s: ,, ,, ,, ,, steel or iron,

N_c: axial stress to be carried by concrete,

N_s: ,, ,, ,, ,, ,, steel or iron,

then,

$$EI=E_cI_c+E_sI_s$$
$$EA=E_cA_c+E_sA_s$$

$$M_c=\frac{E_cI_c}{EI}M,\qquad M_s=\frac{E_sI_s}{EI}M$$

$$N_c=\frac{E_cA_c}{EA}N,\qquad N_s=\frac{E_sA_s}{EA}N$$

$$\left.\begin{aligned}\frac{M_c}{I_c}&=\frac{M}{I_c+\frac{E_s}{E_c}I_s} &\qquad \frac{M_s}{I_s}&=\frac{E_s}{E_c}\cdot\frac{M}{I_c+\frac{E_s}{E_c}I_s}\\ \frac{N_c}{A_c}&=\frac{N}{A_c+\frac{E_s}{E_c}A_s} &\qquad \frac{N_s}{A_s}&=\frac{E_s}{E_c}\cdot\frac{N}{A_c+\frac{E_s}{E_c}A_s}\end{aligned}\right\}\quad\ldots\ldots(15)$$

$\therefore$

$$\left.\begin{aligned}f_c&=\frac{N}{A_c+mA_s}\pm\frac{My}{I_c+mI_s}\\ f_s&=m\left\{\frac{N}{A_c+mA_s}\pm\frac{My}{I_c+mI_s}\right\}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

where

f_c: intensity of stresses in concrete at the point distant y from the neutral axis,

f_s: intensity of stresses in steel or iron at the point distant y from the neutral axis,

$m=\frac{E_s}{E_c}$

$=15$ for steel and concrete,

$=7$ for cast iron and concrete.

The formulas derived so far are quite general and can be applied to the tunnel sections of any form, built of elastic material, when the netural axis of the section is located; and in the following sections, their application to the cases, most commonly met in subaqueous tunnel design, will be explained with some numerical examples.

APPLICATION OF GENERAL EQUATIONS TO THE CIRCULAR TUBE WITH UNIFORM CROSS-SECTION

This is the most common form for shield driven tunnel, and is usually built of cast iron segments and lined afterwards with concrete.

1. Effect of Dead Load and Water Pressure

Let

r: radius of the tube at the neutral axis,

c: depth of water above the neutral axis at the top of the tube,

γ: weight of water per unit volume,

g: weight of the tube per unit area, in the plane of the neutral axis.

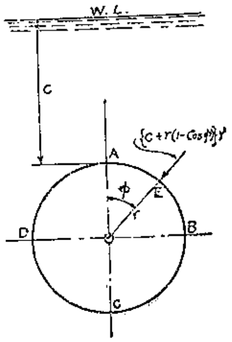


Fig. 3

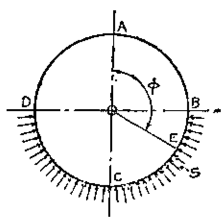


Fig. 4

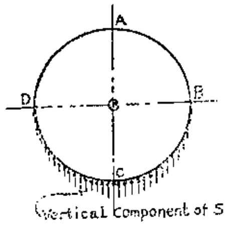


Fig. 5

Intensity of water pressure at any point E making central angle ϕ with the neutral axis is

$$\{c+r(1-\cos\phi)\}\,\gamma.$$

Total upward force of water or buoyancy per unit length of the tube is

$$2\int_0^{\pi}\{c+r(1-\cos\phi)\}\,\gamma.\,rd\phi.\cos\phi=-\pi r^2\gamma.$$

Weight of the tube per unit length is $2\pi rg$.

Case 1. $2\pi rg>\pi r^2\gamma$ or $g-\dfrac{\gamma r}{2}>0$

In this case, weight of the tube in water per unit length is

$$2\pi rg-\pi r^2\gamma=2\pi r\left(g-\frac{\gamma r}{2}\right).$$

This loading must be in equilibrium with the upward reaction acting along the lower half circle of the tube.

As explained before, this reaction is assumed to be of uniform intensity, s per unit area, acting in radial direction, and sum of its vertical components equal to the weight of the tube as weighed in water and as shown in the **Fig. 4** and **5**.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} r d\phi . s . \cos\phi = 2\pi r\left(g-\frac{\gamma r}{2}\right)$$

$$\therefore \qquad s=-\pi\left(g-\frac{\gamma r}{2}\right)$$

Negative sign means that the reaction is on upward direction.

Thus, there are three kinds of external forces which constitute a state of equilibrium; g, weight of the tube acting downward, γ, water pressure acting normal all over the surface of the tube, and s, reaction of earth acting normal on the lower half circle of the tube; and in order to find moment and thrust at the crown point (M_0, H_0), we have to calculate, first, moments of these three kinds of forces.

1. Let

E: any point on the neutral axis making central angle ϕ with the vertical axis,

F: any point between A and E making central angle ξ with the vertical axis,

m_1: moment at E of the weight of the tube between A and E.

Since the weight of unit element of the tube at F is $grd\xi$, the moment of this elementary weight taken at E is

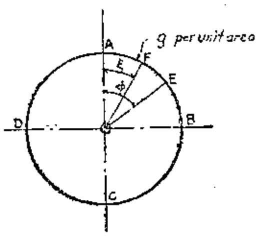


Fig. 6

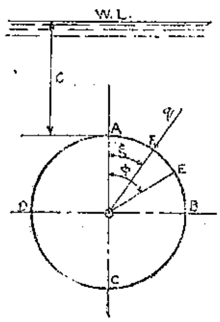


Fig. 7

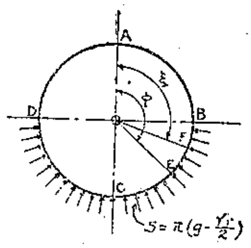


Fig. 8

$$dm_1=-gr^2(\sin\phi-\sin\xi)\,d\xi$$

$$\therefore \qquad m_1=-\int_0^{\phi} gr^2(\sin\phi-\sin\xi)\,d\xi$$

$$=gr^2(1-\cos\phi-\phi\sin\phi)$$

2. Let

m_2: moment at E of the water pressure acting between A and E,

q: intensity of water pressure at the point F,

$$=\{c+r(1-\cos\xi)\}\gamma$$

$$dm_2=-q.\,rd\xi.\,r\sin(\phi-\xi)$$

$\therefore$

$$m_2=-\int_0^{\phi}\gamma r^2\{c+r(1-\cos\xi)\}\sin(\phi-\xi)\,d\xi$$

$$=-\gamma r^2\{c(1-\cos\phi)+r(1-\cos\phi-\tfrac{1}{2}\phi\sin\phi)\}$$

3. Let

m_3: moment at E of the reaction acting between A and E.

$$s=\pi\left(g-\frac{\gamma r}{2}\right)$$

$$dm_3=-\pi\left(g-\frac{\gamma r}{2}\right)r\,d\xi.\,r\sin(\phi-\xi)$$

$$m_3=-\pi\left(g-\frac{\gamma r}{2}\right)r^2\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)\,d\xi$$

$$=-\pi\left(g-\frac{\gamma r}{2}\right)r^2(1-\sin\phi)$$

Now in the equation (12), we have

$$ds=r\,d\phi,\quad a=r,\quad y=r\cos\phi$$

$$\int_0^{\pi}ds=\pi r\qquad\int_0^{\pi}(a-y)\,ds=\pi r^2\qquad\int_0^{\pi}(a-y)^2ds=\frac{3}{2}\pi r^3$$

$$\int_0^{\pi}mds=\int_0^{\pi}m_1ds+\int_0^{\pi}m_2ds+\int_{\frac{\pi}{2}}^{\pi}m_3ds$$

$$\int_0^{\pi}m_1ds=gr^3\int_0^{\pi}(1-\cos\phi-\phi\sin\phi)\,d\phi=0$$

$$\int_0^{\pi}m_2ds=-\gamma r^3\left\{c\int_0^{\pi}(1-\cos\phi)\,d\phi+r\int_0^{\pi}\left(1-\cos\phi-\frac{1}{2}\phi\sin\phi\right)d\phi\right\}$$

$$=-\pi r^3\gamma\left(c+\frac{r}{2}\right)$$

$$\int_{\frac{\pi}{2}}^{\pi}m_3\,ds=-\pi\left(g-\frac{\gamma r}{2}\right)r^3\int_{\frac{\pi}{2}}^{\pi}(1-\sin\phi)\,d\phi$$

$$=-\pi\left(\frac{\pi}{2}-1\right)r^3\left(g-\frac{\gamma r}{2}\right)$$

$$\therefore\quad \int_0^\pi m ds=-\left\{\gamma\left(c+\frac{r}{2}\right)+\left(\frac{\pi}{2}-1\right)\left(g-\frac{\gamma r}{2}\right)\right\}\pi r^3$$

$$\int_0^\pi m(a-y)\,ds=\int_0^\pi m_1(a-y)ds+\int_0^\pi m_2(a-y)ds+\int_{\frac{\pi}{2}}^\pi m_3(a-y)ds$$

$$\int_0^\pi m_1(a-y)\,ds=gr^4\int_0^\pi(1-\cos\phi-\phi\sin\phi)(1-\cos\phi)\,d\phi$$

$$=\frac{\pi}{4}gr^4$$

$$\int_0^\pi m_2(a-y)\,ds=-\gamma r^4\left\{c\int_0^\pi(1-\cos\phi)^2\,d\phi+\right.$$

$$\left.r\int_0^\pi(1-\cos\phi-\frac{1}{2}\phi\sin\phi)(1-\cos\phi)\,d\phi\right\}$$

$$=-\frac{\gamma r^4}{8}\left\{12\pi c+7\pi r\right\}$$

$$\int_{\frac{\pi}{2}}^\pi m_3(a-y)\,ds=-\pi\left(g-\frac{\gamma r}{2}\right)r^4\int_{\frac{\pi}{2}}^\pi(1-\sin\phi)(1-\cos\phi)\,d\phi$$

$$=-\frac{\pi}{2}\left(\pi-1\right)\left(g-\frac{\gamma r}{2}\right)r^4$$

$$\therefore\quad \int_0^\pi m(a-y)ds=\left\{\frac{g}{4}-\frac{12c+7r}{8}\gamma-\frac{\pi-1}{2}\left(g-\frac{\gamma r}{2}\right)\right\}\pi r^4$$

Putting these values in the equation (12), we get

$$M_0=\frac{\pi-3}{2}\cdot r^2\left(g-\frac{\gamma r}{2}\right) \qquad \text{(17)}$$

$$H_0=\frac{r}{2}\left\{(2c+r)\gamma+\left(g-\frac{\gamma r}{2}\right)\right\} \qquad \text{(18)}$$

Moment at any point on the neutral axis in upper half circle of the tube is

$$M=M_0+H_0r(1-\cos\phi)+m_1+m_2$$

$$=r^2\left(g-\frac{\gamma r}{2}\right)\left(\frac{\pi}{2}-\frac{3}{2}\cos\phi-\phi\sin\phi\right) \qquad \text{(19)}$$

Moment at any point on the neutral axis in the lower half circle of the tube is

$$M=M_0+H_0r(1-\cos\phi)+m_1+m_2+m_3$$

$$=r^2\left(g-\frac{\gamma r}{2}\right)\left\{-\frac{\pi}{2}-\frac{3}{2}\cos\phi+(\pi-\phi)\sin\phi\right\} \qquad \text{(20)}$$

Thrust at any point on the neutral axis in the upper half circle of the tube is

$$N=H_0\cos\phi+\int_0^{\phi}g\,r\sin\phi\,d\xi+\gamma r\int_0^{\phi}\{c+r(1-\cos\xi)\}\sin(\phi-\xi)\,d\xi$$

$$=r\gamma\left\{c+r-\frac{r}{2}\cos\phi\right\}+r\left(g-\frac{\gamma r}{2}\right)\left(\frac{1}{2}\cos\phi+\phi\sin\phi\right)\ \ldots\ldots\ldots\ldots(21)$$

Thrust at any point on the neutral axis in the lower half circle of the tube is

$$N=H_0\cos\phi+\int_0^{\phi}gr\sin\phi\,d\xi+\gamma r\int_0^{\phi}\{c+r(1-\cos\xi)\}\sin(\phi-\xi)\,d\xi$$

$$+\pi r\left(g-\frac{\gamma r}{2}\right)\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)\,d\xi$$

$$=r\gamma\left\{c+r-\frac{r}{2}\cos\phi\right\}+r\left(g-\frac{\gamma r}{2}\right)\left\{\frac{1}{2}\cos\phi-(\pi-\phi)\sin\phi+\pi\right\}\ \ldots(22)$$

From (19) and (20), it will be seen that the moment at any point on the neutral axis, due to dead load and water pressure, may be written in the following general form,

$$M_1=r^2\left(g-\frac{\gamma r}{2}\right)F_0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

where

$F_0=\frac{\pi}{2}-\frac{3}{2}\cos\phi-\phi\sin\phi$ .... for the value of ϕ between 0 & $\pi/2$

$F_0=-\frac{\pi}{2}-\frac{3}{2}\cos\phi+(\pi-\phi)\sin\phi$ .. for the value of ϕ between $\pi/2$ & π

Thrust may also be written in the following form,

$$N_1=r\left\{\gamma k+\left(g-\frac{\gamma r}{2}\right)L_0\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

where

$$k=c+r-\frac{r}{2}\cos\phi$$

$L_0=\frac{1}{2}\cos\phi+\phi\sin\phi$ .... for the value of ϕ between 0 & $\pi/2$

$L_0=\frac{1}{2}\cos\phi-(\pi-\phi)\sin\phi+\pi$ .. for the value of ϕ between $\pi/2$ & π

Case 2. $2\pi rg=\pi r^2\gamma$ or $g-\frac{\gamma r}{2}=0$

In this case the tube is in swimming condition, and from the equation

(23), it will be seen that the moment is zero throughout the section; axial stress is given by

$$N=\gamma r\left(c+r-\frac{r}{2}\cos\phi\right) \dots\dots\dots\dots\dots\dots\dots(25)$$

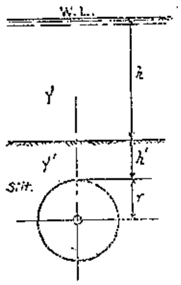


Fig. 9

Very soft ground.

When a tunnel is built in a very soft soil, like silt' which can be supposed to act like a fluid, the equations (23) and (24) can be applied with some modification. Let

γ': weight of silt per unit volume,

h': depth of silt above the neutral axis of the tube,

h: depth of water,

then, substituting γ' for γ, and $h'+h\frac{\gamma}{\gamma'}$ for c, we obtain

$$\left.\begin{aligned} M&=r^2\left(g-\frac{\gamma' r}{2}\right)F_0 \\ N&=r\left\{\gamma\left(r+h'+h\frac{\gamma}{\gamma'}-\frac{1}{2}r\cos\phi\right)+\left(g-\frac{\gamma' r}{2}\right)L_0\right\} \end{aligned}\right\}\dots(26)$$

However, if such a fluidity can be imagined in a silt, the tube can be kept in equilibrium only when $g=\frac{\gamma' r}{2}$, or when the weight of the tube is equal to the weight of the displaced soil.

There will be no moment in the section, for this case, and the axial stress will be given by the equation (26).

Case 3. $2\pi rg<\pi r^2\gamma$ or $g-\frac{\gamma r}{2}<0$

In this case, there is no reaction acting on the bottom surface of the tube, but on the contrary, the tube has the tendency to float up, and can only be kept in the state of equilibrium,

(1) by the weight of overlying earth acting downward on the upper half circle of the section,

(2) by weighing down the section at the inside bottom of the tube.

First, when the buoyancy of the tube is balanced by the weight of overlying earth.

Let

γ': weight of earth per unit volume,

γ : weight of water per unit volume,

γ_1: weight of earth as weighed in water per unit volume,

$=\gamma'-\gamma$.

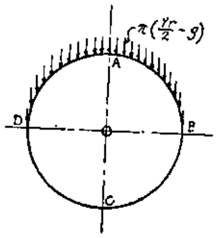


Fig. 10

Depth of earth necessary to overcome buoyancy of the tube is

$$\frac{\pi r\left(\frac{\gamma r}{2}-g\right)}{r\gamma_1}=\frac{\pi}{\gamma_1}\left(\frac{\gamma r}{2}-g\right)$$

and the intensity of this loading is $\pi\left(\frac{\gamma r}{2}-g\right)$ per unit area.

Moment of this load taken on the neutral axis is

$$m_3=-\frac{\pi}{2}\left(\frac{\gamma r}{2}-g\right)r^2\sin^2\phi \text{ ....for the value of } \phi \text{ between } 0 \ \& \ \pi/2$$

$$m_3=\pi\left(\frac{\gamma r}{2}-g\right)r^2\left(\frac{1}{2}-\sin\phi\right) \text{ ......for the value of } \phi \text{ between } \pi/2 \ \& \ \pi$$

$$\int_0^\pi m_3\,ds=-\frac{\pi}{2}\left(\frac{\gamma r}{2}-g\right)r^3\int_0^{\frac{\pi}{2}}\sin^2\phi\,d\phi+\pi\left(\frac{\gamma r}{2}-g\right)r^3\int_{\frac{\pi}{2}}^{\pi}\left(\frac{1}{2}-\sin\phi\right)d\phi$$

$$=\pi\left(\frac{\gamma r}{2}-g\right)r^3\left(\frac{\pi}{8}-1\right)$$

$$\int_0^\pi m_3(a-y)ds=-\frac{\pi}{2}\left(\frac{\gamma r}{2}-g\right)r^4\int_0^{\frac{\pi}{2}}\sin^2\phi(1-\cos\phi)\,d\phi$$

$$+\pi\left(\frac{\gamma r}{2}-g\right)r^4\int_{\frac{\pi}{2}}^{\pi}\left(\frac{1}{2}-\sin\phi\right)(1-\cos\phi)d\phi$$

$$=\pi\left(\frac{\gamma r}{2}-g\right)r^4\left(\frac{\pi}{8}-\frac{5}{6}\right)$$

$$\therefore \quad \int_0^\pi m\,ds=\int_0^\pi m_1 ds+\int_0^\pi m_2 ds+\int_0^\pi m_3 ds$$

$$=\left\{-\gamma\left(c+\frac{r}{2}\right)+\left(\frac{\pi}{8}-1\right)\left(\frac{\gamma r}{2}-g\right)\right\}\pi r^3$$

$$\int_0^\pi m(a-y)ds=\int_0^\pi m_1(a-y)ds+\int_0^\pi m_2(a-y)ds+\int_0^\pi m_3(a-y)ds$$

$$=\left\{\frac{g}{4}-\frac{12c+7r}{8}\gamma+\left(\frac{\pi}{8}-\frac{5}{6}\right)\left(\frac{\gamma r}{2}-g\right)\right\}\pi r^4$$

Hence from (12) we get

$$M_0=\left(\frac{5}{6}-\frac{\pi}{8}\right)r^2\left(\frac{\gamma r}{2}-g\right)\ldots\ldots(27)$$

$$H_0=\frac{r}{6}\left\{(6c+3r)\gamma+\left(\frac{\gamma r}{2}-g\right)\right\}\ldots\ldots(28)$$

Moment at any other point on the neutral axis is

$$M=M_0+H_0r(1-\cos\phi)+m_1+m_2+m_3$$

For $\phi=0$ to $\phi=\frac{\pi}{2}$,

$$M=r^2\left(\frac{\gamma r}{2}-g\right)\left\{-\frac{\pi}{8}+\frac{5}{6}\cos\phi+\phi\sin\phi-\frac{\pi}{2}\sin^2\phi\right\}\ldots(29)$$

For $\phi=\frac{\pi}{2}$ to $\phi=\pi$,

$$M=r^2\left(\frac{\gamma r}{2}-g\right)\left\{\frac{3}{8}\pi+\frac{5}{6}\cos\phi-(\pi-\phi)\sin\phi\right\}\ldots\ldots(30)$$

Thrust at any point on the neutral axis is

For $\phi=0$ to $\phi=\frac{\pi}{2}$,

$$N=H_0\cos\phi+\int_0^\phi gr\sin\phi d\xi+\gamma r\int_0^\phi\{c+r(1-\cos\xi)\}\sin(\phi-\xi)d\xi$$

$$+\pi\left(\frac{\gamma r}{2}-g\right)r\sin\phi\int_0^\phi\cos\xi d\xi$$

$$=\gamma r\left\{c+r-\frac{r}{2}\cos\phi\right\}+r\left(\frac{\gamma r}{2}-g\right)\left\{\frac{1}{6}\cos\phi-\phi\sin\phi+\pi\sin^2\phi\right\}\ldots(31)$$

For $\phi=\frac{\pi}{2}$ to $\phi=\pi$

$$N=H_0\cos\phi+\int_0^\phi gr\sin\phi d\xi+\gamma r\int_0^\phi\{c+r(1-\cos\xi)\}\sin(\phi-\xi)d\xi$$

$$+\pi\left(\frac{\gamma r}{2}-g\right)r\sin\phi$$

$$=\gamma r\left\{c+r-\frac{r}{2}\cos\phi\right\}+r\left(\frac{\gamma r}{2}-g\right)\left\{\frac{1}{6}\cos\phi+(\pi-\phi)\sin\phi\right\}\ldots\ldots(32)$$

These formulas may be written in the following form.

$$M_1'=r^2\left(\frac{\gamma r}{2}-g\right)F_0'\ldots\ldots(33)$$

where

$$F_0' = -\frac{\pi}{8} + \frac{5}{6}\cos\phi + \phi\sin\phi - \frac{\pi}{2}\sin^2\phi \ldots\ldots \text{for the value of } \phi \text{ bet. } 0 \ \& \ \pi/2$$

$$F_0' = \frac{3}{8}\pi + \frac{5}{6}\cos\phi - (\pi - \phi)\sin\phi \ldots\ldots \text{for the value of } \phi \text{ bet. } \pi/2 \ \& \ \pi$$

$$N_1' = r\left\{\gamma k + \left(\frac{\gamma r}{2} - g\right)L_0'\right\} \ldots\ldots (34)$$

where

$$k = c + r\left(1 - \frac{1}{2}\cos\phi\right)$$

$$L_0' = \frac{1}{6}\cos\phi - \phi\sin\phi + \pi\sin^2\phi \ldots\ldots \text{for the value of } \phi \text{ bet. } 0 \ \& \ \pi/2.$$

$$L_0' = \frac{1}{6}\cos\phi + (\pi - \phi)\sin\phi \ldots\ldots \text{for the value of } \phi \text{ bet. } \pi/2 \ \& \ \pi.$$

In this case, it should be remembered in the calculation of moment and thrust due to the earth load, treated in the next article, that the depth of earth cover shall be reduced by $\frac{\pi}{\gamma_1}\left(\frac{\gamma r}{2} - g\right)$.

Second, when the tube is prevented from floating up by weighing down at its bottom point, before concrete lining is put on, the amount of the load necessary to overcome buoyancy will be $2\pi r\left(\frac{\gamma r}{2} - g\right)$ per unit length of the tube.

This load, being supposed to be applied at the point C of the tube, will contribute nothing in the calculation of the moment of external force, m, or $m_3 = 0$.

Hence we have

$$\int_0^\pi m ds = -\pi r^3 \gamma\left(c + \frac{r}{2}\right)$$

$$\int_0^\pi m(a - y)ds = \left(\frac{g}{4} - \frac{12c + 7r}{8}\gamma\right)\pi r^4$$

and from (12)

$$M_0 = -\frac{r^2}{2}\left(\frac{\gamma r}{2} - g\right) \ldots\ldots (35)$$

$$H_0 = \frac{r}{2}\left\{(2c + r)\gamma + \left(\frac{\gamma r}{2} - g\right)\right\} \ldots\ldots (36)$$

Moment at any point on the neutral axis is

$$M=M_0+H_0r(1-\cos\phi)+m_1+m_2$$
$$=r^2\left(\frac{\gamma r}{2}-g\right)\left(-1+\frac{1}{2}\cos\phi+\phi\sin\phi\right)\dots\dots(37)$$

Thrust at any point on the neutral axis is

$$N=H_0\cos\phi+\int_0^\phi gr\sin\phi d\xi+\gamma r\int_0^\phi\{c+r(1-\cos\phi)\}\sin(\phi-\xi)d\xi$$
$$=\gamma r\left\{c+r\left(1-\frac{1}{2}\cos\phi\right)\right\}+r\left(\frac{\gamma r}{2}-g\right)\left(\frac{1}{2}\cos\phi-\phi\sin\phi\right)\dots\dots(38)$$

or in general,

$$M_1''=r^2\left(\frac{\gamma r}{2}-g\right)F_0''\dots\dots(39)$$

where $F_0''=-1+\dfrac{1}{2}\cos\phi+\phi\sin\phi$

$$N_1''=r\left\{\gamma k+r\left(\frac{\gamma r}{2}-g\right)L_0''\right\}\dots\dots(40)$$

where $L_0''=\dfrac{1}{2}\cos\phi-\phi\sin\phi$

2. Effect of Earth Load overlying the Tube

Let

h : height of earth cover above the neutral axis at the crown of the tube,

γ : weight of water per unit volume,

γ' : weight of earth per unit volume,

γ_1 : weight of earth in water per unit volume $=\gamma'-\gamma$.

Total weight of submerged earth acting on the tube,

$$W=r\gamma_1\left\{2(h+r)-\frac{\pi r}{2}\right\}$$

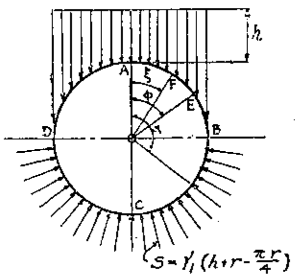


Fig. 11

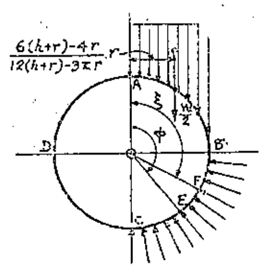


Fig. 12

If the intensity of normal reaction to balance this load is s per unit area,

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} r.\, d\alpha.\, s.\cos\alpha = r\gamma_1\left\{2(h+r)-\frac{\pi r}{2}\right\}$$

$$\therefore \qquad s=-\gamma_1\left\{h+r-\frac{\pi r}{4}\right\}$$

Moment of external forces taken at any point between A and B.

Let E and F be any point on the neutral axis with central angles ϕ and ξ as shown in the figure.

Then the moment of the earth load acting on the elementary area at F taken about the point E is

$$dm_1=-\gamma_1\{h+r(1-\cos\xi)\}\, r.\, d\xi.\cos\xi.\, r(\sin\phi-\sin\xi)$$

$$m_1=\int_0^{\phi} dm_1$$

$$=-\frac{\gamma_1 r^2}{6}\{3(h+r)\sin^2\phi-3r\phi\sin\phi-r\sin^2\phi\cos\phi+2r(1-\cos\phi)\}$$

Moment of external forces taken at any point between B and C. By putting $\phi=\frac{\pi}{2}$ in the above equation, we get

$$m_1 \text{ at } B=-\frac{\gamma_1 r^3}{12}\{6(h+r)-(3\pi-4)r\}$$

Distance of the center of gravity of the earth load from the point B is

$$\frac{\dfrac{\gamma_1 r^2}{12}\{6(h+r)-3\pi r+4r\}}{\dfrac{1}{2}\gamma_1 r\left\{2(h+r)-\dfrac{\pi r}{2}\right\}}=\frac{6(h+r)-3\pi r+4r}{12(h+r)-3\pi r}r$$

Horizontal distance of $\frac{W}{2}$ from the axis OA is

$$\frac{6(h+r)-4r}{12(h+r)-3\pi r}r$$

Hence the moment of external forces at any point E between B and C is

$$m_2=\frac{1}{2}\gamma_1 r\left\{2(h+r)-\frac{\pi r}{2}\right\}.\left\{\frac{6(h+r)-4r}{12(h+r)-3\pi r}r-r\sin\phi\right\}$$
$$-\gamma_1\left\{(h+r)-\frac{\pi r}{4}\right\}r^2\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)d\xi$$

$$=-\frac{\gamma_1 r^2}{12}\{6(h+r)-(3\pi-4)r\}$$

$$\int_0^\pi m ds = \int_0^{\frac{\pi}{2}} m_1 ds + \int_{\frac{\pi}{2}}^\pi m_2 ds = r\left\{\int_0^{\frac{\pi}{2}} m_1 d\phi + \int_{\frac{\pi}{2}}^\pi m_2 d\phi\right\}$$

$$= -\frac{\gamma_1 r^3}{72}\{27\pi h - (9\pi^2 - 51\pi + 64)r\}$$

$$\int_0^\pi m(a-y)ds = r^2\left\{\int_0^{\frac{\pi}{2}} m_1(1-\cos\phi)d\phi + \int_{\frac{\pi}{2}}^\pi m_2(1-\cos\phi)d\phi\right\}$$

$$= -\frac{\gamma_1 r^4}{288}\{(108\pi + 96)h - (36\pi^2 - 177\pi + 160)r\}$$

Hence, from (12) we get

$$M_0 = \frac{\gamma_1 r^3}{144\pi}\{(54\pi - 96)(h+r) - (18\pi^2 - 75\pi + 128)r\} \quad \ldots\ldots\ldots (41)$$

$$H_0 = \frac{\gamma_1 r}{48\pi}\{32(h+r) - 9\pi r\} \quad \ldots\ldots\ldots (42)$$

Moment at any point on the neutral axis in the upper half circle of the tube is

$$M = M_0 + H_0 r(1-\cos\phi) + m_1$$

$$= \gamma_1 r^3\left\{\frac{1}{6}\sin^2\phi\cos\phi + \frac{1}{2}\phi\sin\phi + \frac{25}{48}\cos\phi - 0.67564 + \frac{h+r}{r}\left(\frac{3}{8} - \frac{2}{3\pi}\cos\phi - \frac{1}{2}\sin^2\phi\right)\right\} \quad \ldots (43)$$

Moment at any point on the neutral axis in the lower half circle of the tube is

$$M = M_0 + H_0 r(1-\cos\phi) + m_2$$

$$= \gamma_1 r^3\left\{\frac{3}{16}\cos\phi + 0.10976 - \frac{h+r}{r}\left(\frac{2}{3\pi}\cos\phi + \frac{1}{8}\right)\right\} \quad \ldots\ldots (44)$$

In general,

$$M_2 = \gamma_1 r^3\{F_1 + nF_2\} \quad \ldots\ldots\ldots (45)$$

where

$$n = \frac{h+r}{r}$$

$$\left.\begin{aligned} F_1 &= \frac{1}{6}\sin^2\phi\cos\phi + \frac{1}{2}\phi\sin\phi + \frac{25}{48}\cos\phi - 0.67564 \\ F_2 &= \frac{3}{8} - \frac{2}{3\pi}\cos\phi - \frac{1}{2}\sin^2\phi \quad \ldots\ldots \end{aligned}\right\} \text{ for the value of } \phi \text{ between } 0 \text{ \& } \frac{\pi}{2}$$

$$\left.\begin{aligned} F_1 &= \frac{3}{16}\cos\phi + 0.10976 \\ F_2 &= -\frac{2}{3\pi}\cos\phi - \frac{1}{8} \end{aligned}\right\} \text{for the value of } \phi \text{ between } \frac{\pi}{2} \ \& \ \pi$$

Thrust at any point on the neutral axis in the upper half circle is

$$N = H_0\cos\phi + \gamma_1 r\int_0^\phi \{(h+r) - r\cos\xi\}\cos\xi.\,d\xi.\,\sin\phi$$

$$= \gamma_1 r^2\left\{\left(\frac{2}{3\pi}\cos\phi + \sin^2\phi\right)\frac{h+r}{r} - \left(\frac{3}{16}\cos\phi + \frac{1}{2}\phi\sin\phi + \frac{1}{2}\sin^2\phi\cos\phi\right)\right\} \quad \text{(46)}$$

Thrust at any point on the neutral axis in the lower half circle is

$$N = H_0\cos\phi + \gamma_1 r\left\{(h+r) - \frac{\pi r}{4}\right\}\sin\phi + \gamma_1\left\{(h+r) - \frac{\pi r}{4}\right\} r\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)d\xi$$

$$= \gamma_1 r^2\left\{\left(\frac{2}{3\pi}\cos\phi + 1\right)\frac{h+r}{r} - \frac{\pi}{4} - \frac{3}{16}\cos\phi\right\} \quad \text{(47)}$$

In general,

$$N_2 = \gamma_1 r^2\{nL_2 - L_1\} \quad \text{(48)}$$

where

$$n = \frac{h+r}{r}$$

$$\left.\begin{aligned} L_1 &= \frac{3}{16}\cos\phi + \frac{1}{2}\phi\sin\phi + \frac{1}{2}\sin^2\phi\cos\phi \\ L_2 &= \frac{2}{3\pi}\cos\phi + \sin^2\phi \end{aligned}\right\} \text{for the value of } \phi \text{ between } 0 \ \& \ \frac{\pi}{2}$$

$$\left.\begin{aligned} L_1 &= \frac{3}{16}\cos\phi + \frac{\pi}{4} \\ L_2 &= \frac{2}{3\pi}\cos\phi + 1 \end{aligned}\right\} \text{for the value of } \phi \text{ between } \frac{\pi}{2} \ \& \ \pi$$

3. Effect of Superimposed Uniform Loading

In some case, it will be necessary to check the strength of the tube due to the loading that may be superimposed over the tunnel from sunken vessels. This loading is assumed to be of uniform intensity over the horizontal projection of the tube, w per unit area.

If the intensity of normal reaction is s per unit area,

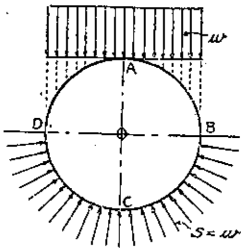


Fig. 13

$$\int_{\frac{\pi}{2}}^{\pi} r\,d\phi\,.\,s\cos\phi = wr$$

$$\therefore \quad s = -w$$

Moment of external forces taken at any point on neutral axis bet. A and B.

$$m_1 = -\frac{1}{2}\,wr^2\sin^2\phi$$

Moment of external forces taken at any point bet. B and C.

$$m_2 = wr\left(\frac{r}{2} - r\sin\phi\right) - wr^2\int_{\frac{\pi}{2}}^{\pi}\sin(\phi-\xi)\,d\xi$$

$$= -\frac{1}{2}\,wr^2$$

$$\int_0^{\pi} m\,ds = \int_0^{\frac{\pi}{2}} m_1\,ds + \int_{\frac{\pi}{2}}^{\pi} m_2\,ds = -\frac{3}{8}\,w\pi r^3$$

$$\int_0^{\pi} m(a-y)\,ds = \int_0^{\frac{\pi}{2}} m_1(a-y)\,ds + \int_{\frac{\pi}{2}}^{\pi} m_2(a-y)\,ds$$

$$= -\frac{9\pi+8}{24}\,wr^4$$

Hence, from (12) we get

$$M_0 = \frac{9\pi-16}{24\pi}\,.\,wr^2 \quad \cdots\cdots (49)$$

$$H_0 = \frac{2wr}{3\pi} \quad \cdots\cdots (50)$$

Moment at any point on the neutral axis bet. A and B is

$$M = M_0 + H_0 r(1-\cos\phi) + \frac{1}{2}\,wr^2\sin^2\phi$$

$$= wr^2\left\{\frac{3}{8} - \frac{2}{3\pi}\cos\phi - \frac{1}{2}\sin^2\phi\right\} \quad \cdots\cdots (51)$$

Moment at any point on the neutral axis bet. B and C is

$$M = M_0 + H_0 r(1-\cos\phi) - \frac{1}{2}\,wr^2$$

$$= wr^2\left\{-\frac{2}{3\pi}\cos\phi - \frac{1}{8}\right\} \quad \cdots\cdots (52)$$

Thrust at any point on the neutral axis bet. A and B is

$$N = H_0 \cos\phi + wr \sin\phi \int_0^{\phi} \cos\xi \, d\xi$$

$$= wr\left\{\frac{2}{3\pi}\cos\phi + \sin^2\phi\right\} \quad \cdots\cdots (53)$$

Thrust at any point on the neutral axis bet. B and C is

$$N = H_0 \cos\phi + wr \sin\phi + wr \int_{\frac{\pi}{2}}^{\phi} \sin(\phi - \xi)\, d\xi$$

$$= wr\left(\frac{2}{3\pi}\cos\phi + 1\right) \quad \cdots\cdots (54)$$

In general,

$$M_3 = wr^2 F_2 \quad \cdots\cdots (55)$$

where

$F_2 = \frac{3}{8} - \frac{2}{3\pi}\cos\phi - \frac{1}{2}\sin^2\phi$ ....for the value of ϕ bet. 0 & $\pi/2$

$F_2 = -\frac{2}{3\pi}\cos\phi - \frac{1}{8}$ ....for the value of ϕ bet. $\pi/2$ & π

$$N_3 = wrL_2 \quad \cdots\cdots (56)$$

where

$L_2 = \frac{2}{3\pi}\cos\phi + \sin^2\phi$....for the value of ϕ bet. 0 & $\pi/2$

$L_2 = \frac{2}{3\pi}\cos\phi + 1$ ......for the value of ϕ bet. $\pi/2$ & π

4. Effect of Internal Pressure

Effect of internal pressure due to the use of compressed air during construction will be considered below.

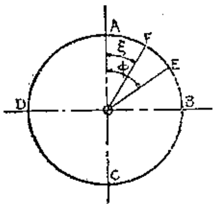


Fig. 14

Let

p: intensity of internal pressure above normal, then, the moment of internal force on the elementary area at F about any point E on the neutral axis is

$$dm = pr^2 \sin(\phi - \xi)\, d\xi$$

$$m = \int_0^{\phi} pr^2 \sin(\phi - \xi)\, d\xi = pr^2(1 - \cos\phi)$$

$$\int_0^{\pi} m\, ds = pr^3 \int_0^{\pi} (1 - \cos\phi)\, d\phi$$

$$= \pi pr^3$$

$$\int_0^{\pi} m(a-y)\,ds = pr^4\int_0^{\pi}(1-\cos\phi)^2\,d\phi$$
$$=\frac{3}{2}\pi pr^4$$

From (12) we get

$$M_0=0 \qquad (57)$$
$$H_0=-pr \qquad (58)$$

Moment at any point on the neutral axis of the tube is

$$M_4=M_0+H_0r(1-\cos\phi)+m=0 \qquad (59)$$

Thrust at any point on the neutral axis of the tube is

$$N_4=H_0\cos\phi-pr\int_0^{\phi}\sin(\phi-\xi)\,d\xi$$
$$=-pr \qquad (60)$$

Hence the internal pressure of the intensity p per unit area will give the section uniform axial tension pr.

5. Effect of Live Load

Effect of live load to be applied at the inside of the tube will now be considered. Usually this effect is very small, and it will serve to reduce the maximum positive moment that will be produced at the crown point of the tube.

Two cases of live load, concentrated and distributed, will be considered.

Case 1. Let W be the live load per unit length of the tube, acting symmetrically about the center line and at a distance b as shown in the figure.

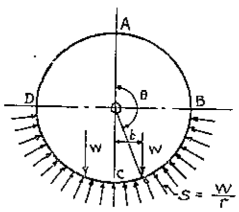


Fig. 15

Then the intensity of uniform normal reaction on the lower half surface of the tube will be $s=-\frac{W}{r}$.

Moment of external forces taken at any point on the neutral axis are

$m=0$ between $\phi=0$ & $\phi=\pi/2$,

$$m=-\frac{W}{r}\cdot r^2\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)\,d\xi=-Wr(1-\sin\phi)$$

between $\phi=\pi/2$ & $\phi=\theta$,

$$m=Wr(\sin\theta-\sin\phi)-Wr\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)\,d\xi$$

$$= Wr(\sin\theta - 1)$$

between $\phi=\theta$ & $\phi=\pi$

$$\therefore \qquad \int_0^\pi m\,ds = Wr^2\left\{-\int_{\frac{\pi}{2}}^{\theta}(1-\sin\phi)\,d\phi + \int_\theta^\pi(\sin\theta - 1)\,d\phi\right\}$$

$$= Wr^2\left\{(\pi-\theta)\sin\theta - \cos\theta - \frac{\pi}{2}\right\}$$

$$\int_0^\pi m(a-y)\,ds = Wr^3\left\{-\int_{\frac{\pi}{2}}^{\theta}(1-\sin\phi)(1-\cos\phi)d\phi + \int_\theta^\pi(\sin\theta-1)(1-\cos\phi)\,d\phi\right\}$$

$$= Wr^3\left\{\frac{1}{2}\sin^2\theta + (\pi-\theta)\sin\theta - \cos\theta - \frac{\pi+1}{2}\right\}$$

Hence, we get from (12)

$$M_0 = \frac{Wr}{\pi}\left\{\frac{\pi}{2} + \cos\theta - (\pi-\theta)\sin\theta - \cos^2\theta\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(61)$$

$$H_0 = \frac{W}{\pi}\cos^2\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(62)$$

Moment at any point on the neutral axis is given by the following equations,

for $\phi=0$ to $\phi=\frac{\pi}{2}$

$$M'_5 = M_0 + H_0 r(1-\cos\phi)$$

$$= \frac{Wr}{\pi}\left\{\frac{\pi}{2} + \cos\theta - (\pi-\theta)\sin\theta - \cos^2\theta\cos\phi\right\} \quad\ldots\ldots\ldots(63)$$

for $\phi=\frac{\pi}{2}$ to $\phi=\theta$

$$M_5'' = M_0 + H_0 r(1-\cos\phi) - Wr(1-\sin\phi)$$

$$= \frac{Wr}{\pi}\left\{-\frac{\pi}{2} + \cos\theta - (\pi-\theta)\sin\theta - \cos^2\theta\cos\phi + \pi\sin\phi\right\}(64)$$

for $\phi=\theta$ to $\phi=\pi$

$$M_5''' = M_0 + H_0 r(1-\cos\phi) - Wr(1-\sin\theta)$$

$$= \frac{W}{\pi}\left\{-\frac{\pi}{2} + \cos\theta + \theta\sin\theta - \cos^2\theta\cos\phi\right\} \quad\ldots\ldots\ldots\ldots(65)$$

Thrust at any point on the neutral axis is given by the following equations,

for $\phi=0$ to $\phi=\frac{\pi}{2}$

$$N_5' = H_0\cos\phi = \frac{W}{\pi}\cos^2\theta\cos\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(66)$$

for $\phi=\frac{\pi}{2}$ to $\phi=\theta$

$$N_5''=H_0\cos\phi+W\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)\,d\xi$$

$$=\frac{W}{\pi}\left\{\pi+\cos^2\theta\cos\phi-\pi\sin\phi\right\} \quad\cdots\cdots\cdots\cdots(67)$$

for $\phi=\theta$ to $\phi=\pi$

$$N_5'''=H_0\cos\phi+W\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)\,d\xi+W\sin\phi$$

$$=\frac{W}{\pi}\left\{\cos^2\theta\cos\phi+\pi\right\} \quad\cdots\cdots\cdots\cdots(68)$$

Case 2. For the distributed live load, let it be of uniform intensity, v per unit area, acting over a horizontal distance b on both sides of the center line.

Reaction to this live load will be, as before,

$$s=-\frac{vb}{r}$$

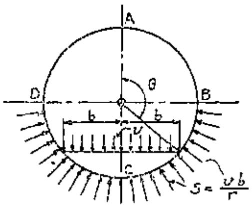


Fig. 16

Moment of external forces taken at any point on the neutral axis is

$$m=0 \qquad \text{between } \phi=0 \;\&\; \phi=\frac{\pi}{2},$$

$$m=-v\,b\,r\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)\,d\xi=-vr^2\sin\theta(1-\sin\phi)$$

between $\phi=\frac{\pi}{2}$ & $\phi=\theta$,

$$m=v\,\frac{(b-r\sin\phi)^2}{2}-v\,b\,r\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)\,d\xi$$

$$=\frac{vr^2}{2}\left\{\sin^2\theta-2\sin\theta+\sin^2\phi\right\}$$

between $\phi=\theta$ & $\phi=\pi$

$$\therefore\quad \int_0^{\pi}m\,ds=vr^3\left\{-\sin\theta\int_{\frac{\pi}{2}}^{\theta}(1-\sin\phi)\,d\phi+\frac{1}{2}\int_{\theta}^{\pi}(\sin^2\theta-2\sin\theta+\sin^2\phi)\,d\phi\right\}$$

$$=vr^3\left\{\frac{\pi-\theta}{2}\sin^2\theta-\frac{\pi}{2}\sin\theta-\frac{3}{4}\sin\theta\cos\theta+\frac{\pi-\theta}{4}\right\}$$

$$\int_0^{\pi}m(a-y)\,ds=vr^4\left\{-\sin\theta\int_{\frac{\pi}{2}}^{\theta}(1-\sin\phi)(1-\cos\phi)\,d\phi\right.$$

$$\left.+\frac{1}{2}\int_{\theta}^{\pi}(\sin^2\theta-2\sin\theta+\sin^2\phi)(1-\cos\phi)\,d\phi\right\}$$

$$=vr^4\left\{\frac{\pi-\theta}{4}-\frac{\pi+1}{2}\sin\theta+\frac{\pi-\theta}{2}\sin^2\theta+\frac{1}{6}\sin^3\theta-\frac{3}{4}\sin\theta\cos\theta\right\}$$

Hence from (12), we get

$$M_0=\frac{vr^2}{\pi}\left\{\frac{3}{4}\sin\theta\cos\theta+\frac{1}{3}\sin^3\theta-\frac{\pi-\theta}{2}\sin^2\theta+\frac{\pi-\theta}{2}\sin\theta-\frac{\pi-\theta}{4}\right\} \quad \ldots\ldots(69)$$

$$H_0=\frac{vr}{\pi}\sin\theta\left(1-\frac{1}{3}\sin^2\theta\right) \quad \ldots\ldots(70)$$

Moment at any point on the neutral axis is given by the following equations.

For $\phi=0$ to $\phi=\frac{\pi}{2}$

$$M'_6=M_0+H_0r(1-\cos\phi)$$
$$=\frac{vr^2}{\pi}\left\{\frac{3}{4}\sin\theta\cos\theta-\frac{\pi-\theta}{2}\sin^2\theta+\frac{\pi-\theta+2}{2}\sin\theta-\frac{\pi-\theta}{4}\right.$$
$$\left.-\sin\theta\left(1-\frac{1}{3}\sin^2\theta\right)\cos\phi\right\} \quad \ldots(71)$$

For $\phi=\frac{\pi}{2}$ to $\phi=0$

$$M_6''=M_0+H_0r(1-\cos\phi)-\frac{vr^2}{\pi}\pi\sin\theta(1-\sin\phi)$$
$$=\frac{vr^2}{\pi}\left\{\frac{3}{4}\sin\theta\cos\theta-\frac{\pi-\theta}{2}\sin^2\theta-\frac{\pi+\theta-2}{2}\sin\theta-\frac{\pi-\theta}{4}\right.$$
$$\left.-\pi\sin\theta\sin\phi-\sin\theta\left(1-\frac{1}{3}\sin^2\theta\right)\cos\phi\right\} \quad \ldots(72)$$

For $\phi=0$ to $\phi=\pi$

$$M_6'''=M_0+H_0r(1-\cos\phi)+\frac{vr^2}{\pi}\frac{\pi}{2}(\sin^2\theta-2\sin\theta+\sin^2\phi)$$
$$=\frac{vr^2}{\pi}\left\{\frac{3}{4}\sin\theta\cos\theta+\frac{\theta}{2}\sin^2\theta-\frac{\pi+\theta-2}{2}\sin\theta-\frac{\pi-\theta}{4}\right.$$
$$\left.+\frac{\pi}{2}\sin^2\phi-\sin\theta\left(1-\frac{1}{3}\sin^2\theta\right)\cos\phi\right\} \quad \ldots\ldots(73)$$

Thrust at any point on the neutral axis is given by the following equations.

For $\phi=0$ to $\phi=\frac{\pi}{2}$

$$N_6'=H_0\cos\phi=\frac{vr}{\pi}\sin\theta\left(1-\frac{1}{3}\sin^2\theta\right)\cos\phi \quad \ldots\ldots(74)$$

For $\phi=\frac{\pi}{2}$ to $\phi=\theta$

$$N_0''=H_0\cos\phi+v\,b\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)\,d\xi$$

$$=\frac{vr}{\pi}\sin\theta\left\{\pi(1-\sin\phi)+\left(1-\frac{1}{3}\sin^2\theta\right)\cos\phi\right\}\,\ldots\ldots\ldots\ldots\ldots(75)$$

For $\phi=\theta$ to $\phi=\pi$

$$N_0'''=H_0\cos\phi+v\,b\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)\,d\xi+vr(\sin\theta-\sin\phi)\sin\phi$$

$$=\frac{vr}{\pi}\left\{\pi\sin\theta-\pi\sin^2\phi+\sin\theta\left(1-\frac{1}{3}\sin^2\theta\right)\cos\phi\right\}\,\ldots\ldots\ldots\ldots(76)$$

6. Deflection of Elastic Circular Tube

Let

M: Bending moment at any point on the neutral axis of the circular tube produced by given loading,

M_u: Bending moment at the same point due to the load unity applied at the crown point,

then, the deflection of the crown point is given by

$$\delta=\int_0^{2\pi}\frac{M.M_u.ds}{EI}=2\int_0^{\pi}\frac{M.M_u.ds}{EI}$$

If the whole circle is divided into $2n$ equal parts of the length Δs, the above formula is written in the following form.

$$\delta=2\frac{\Delta s}{EI}\sum_1^n M.M_u.\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(77)$$

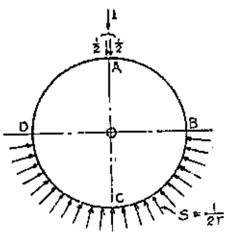


Fig. 17

M, the moment at the center of each division due to any given loading, is determined from the formulas already found, and M_u, is to be found by the following formula.

Let the load unity be applied on the tube as shown in the figure.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} r\,d\phi\,s\cos\phi=1 \quad \therefore \quad s=-\frac{1}{2r}$$

Moment at any point on the neutral axis between A and B is

$$m=-\frac{1}{2}r\sin\phi$$

Moment at any point of the neutral axis bet. B & C is

$$m=-\frac{1}{2}r\sin\phi-\frac{1}{2r}\cdot r^2\int_{\frac{\pi}{2}}^{\phi}\sin(\phi-\xi)d\xi=-\frac{1}{2}r$$

$$\therefore \qquad \int_0^{\pi} m\,ds=-\frac{1}{2}r^2\int_0^{\frac{\pi}{2}}\sin\phi\,d\phi-\frac{1}{2}\cdot r^2\int_{\frac{\pi}{2}}^{\pi}d\phi$$

$$=-\frac{1}{4}r^2(\pi+2)$$

$$\int_0^{\pi} m(a-y)ds=-\frac{1}{2}r^3\int_0^{\frac{\pi}{2}}\sin\phi\,(1-\cos\phi)\,d\phi-\frac{1}{2}\cdot r^3\int_{\frac{\pi}{2}}^{\pi}(1-\cos\phi)d\phi$$

$$=-\frac{1}{4}r^3(\pi+3)$$

Hence from (12), we get

$$M_0=\frac{r}{4} \quad \dots\dots (78)$$

$$H_0=\frac{1}{2\pi} \quad \dots\dots (79)$$

Moment at any point on the neutral axis of the tube is

$$M_u=M_0+H_0 r(1-\cos\phi)+m$$

For $\phi=0$ to $\phi=\frac{\pi}{2}$

$$M_u=r\left\{\frac{1}{4}+\frac{1}{2\pi}-\frac{1}{2\pi}\cos\phi-\frac{1}{2}\sin\phi\right\}$$

For $\phi=\frac{\pi}{2}$ to $\phi=\pi$

$$M_u=r\left\{-\frac{1}{4}+\frac{1}{2\pi}-\frac{\cos\phi}{2\pi}\right\}$$

In general,

$$M_u=r\,F_3 \quad \dots\dots (80)$$

where $F_3=.4092-\frac{1}{2\pi}\cos\phi-\frac{1}{2}\sin\phi$ ....for the value of ϕ bet. 0 & $\pi/2$,

$F_3=.0908-\frac{1}{2\pi}\cos\phi$ ....for the value of ϕ bet. $\pi/2$ & π.

7. Miscellaneous Remerks

For convenience of practical purpose, the coefficients given in the foregoing formulas are calculated for the value of ϕ at every ten degrees and the result are given in the **Tables I, II, III,** and **IV.**

In order to show the effect of several loadings at a glance, diagrams are prepared plotting the coefficients given above on the normals at their corresponding points, positive values being taken inside and negative values outside of the neutral axis. (**Fig. 18~23**)

Maximum stress in the lining occurs at the point of maximum bending moment.

Maximum positive moment is usually produced at the crown point of the tube.

Point of maximum negative moment is different according to the nature of loading, for instance, the value of ϕ for maximum negative moment being 66°47′ for M_1, 84°15′ for M_1' and 77°45′ for M_2 and M_3; but for all practical purpose, the value at the point $\phi=80°$ will be close enough for the maximum negative moment.

It will be seen from the tables of coefficients that the moments M_1, M_1', M_2, M_3 have generally same sign as that of M_u and it means that the deflection is of positive value, while M_1'', and also M_5 and M_6 have different signs with M_u and the deflection will be of negative value, in other word, the vertical diameter of the tube will elongate for such a loading.

Table 1.

ϕ	F_0	F_0'	F_0''	F_1	F_2	F_3
0°	.0708	.4496	−.5000	−.1548	.1628	.2500
10°	.0633	.4109	−.4773	−.1426	.1509	.1657
20°	.0419	.3260	−.4108	−.1082	.1171	.0886
30°	.0100	.1981	−.3052	−.0575	.0662	.0217
40°	−.0271	.0455	−.1682	.0006	.0058	−.0341
50°	−.0619	−.1103	−.0101	.0564	−.0543	−.0761
60°	−.0861	−.2472	.1569	.1008	−.1061	−.1034
70°	−.0903	−.3466	.3191	.1268	−.1391	−.1150
80°	−.0647	−.3964	.4618	.1304	−.1468	−.1108
90°	0	−.3927	.5708	.1098	−.1259	−.0908
100°	.0647	−.3416	.6320	.0772	−.0881	−.0632
110°	.0903	−.2550	.6331	.0457	−.0524	−.0364
120°	.0861	−.1455	.5638	.0160	−.0189	−.0112
130°	.0619	−.0261	.4167	−.0107	.0114	.0115

ϕ	F_0	F_0'	F_0''	F_1	F_2	F_3
140°	.0271	.0909	.1876	−.0338	.0376	.0311
150°	−.0100	.1946	−.1240	−.0526	0.588	.0470
160°	−.0419	.2756	−.5147	−.0664	.0744	.0588
170°	−.0633	.3271	−.9772	−.0749	.0840	.0659
180°	−.0708	.3448	−1.5000	−.0777	.0872	.0684

Table II.

ϕ	$F_1+n\,F_2$			
	$n=1$	$n=2$	$n=3$	$n=4$
0°	.0080	.1708	.3336	.4964
10°	.0083	.1592	.3101	.4610
20°	.0089	.1260	.2431	.3602
30°	.0087	.0749	.1411	.2073
40°	.0064	.0122	.0180	.0238
50°	.0016	−.0532	−.1080	−.1628
60°	−.0053	−.1114	−.2175	−.3236
70°	−.0122	−.1513	−.2904	−.4295
80°	−.0164	−.1632	−.3100	−.4568
90°	−.0152	−.1402	−.2652	−.3902
100°	−.0109	−.0990	−.1871	−.2752
110°	−.0067	−.0591	−.1115	−.1639
120°	−.0029	−.0218	−.0407	−.0596
130°	.0007	.0121	.0235	.0349
140°	.0038	.0414	.0790	.1166
150°	.0062	.0650	.1238	.1826
160°	.0080	.0824	.1568	.2312
170°	.0091	.0931	.1771	.2611
180°	.0095	.0967	.1839	.2711

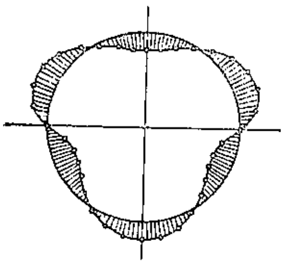
Table III.

ϕ	k	L_0	L_0'	L_0''	L_1	L_2
0°	$c+.5r$	.5000	.1667	.5000	.1875	.2122
10°	$c+.5076r$	.5227	.2285	.4621	.2148	.2392
20°	$c+.5302r$	.5892	.4047	.3504	.2909	.3164
30°	$c+.5670r$	.6948	.6679	.1712	.4016	.4338
40°	$c+.6170r$	.8318	.9769	−.0658	.5263	.5758
50°	$c+.6786r$	.9899	1.2822	−.3471	.6434	.7232
60°	$c+.75r$	1.1569	1.5326	−.6569	.7348	.8561
70°	$c+.8290r$	1.3191	1.6830	−.9771	.7891	.9556
80°	$c+.9132r$	1.4618	1.7008	−1.2882	.8043	1.0067

ϕ	k	L_0	L_0'	L_0''	L_1	L_2
90°	$c+r$	1.5708	1.5708	−1.5708	.7854	1.0000
100°	$c+1.0868r$	1.6798	1.3461	−1.8056	.7528	.9631
110°	$c+1.1710r$	1.8225	1.0911	−1.9751	.7213	.9274
120°	$c+1.25r$	1.9847	.8236	−2.0638	.6916	.8939
130°	$c+1.3214r$	2.1517	.5614	−2.0595	.6649	.8636
140°	$c+1.3830r$	2.3089	.3211	−1.9536	.6418	.8374
150°	$c+1.4330r$	2.4468	.1175	−1.7420	.6230	.8162
160°	$c+1.4698r$	2.5524	−.0372	−1.4249	.6092	.8006
170°	$c+1.4924r$	2.6189	−.1338	−1.0076	.6007	.7910
180°	$c+1.5r$	2.6416	−.1667	−.5000	.5979	.7878

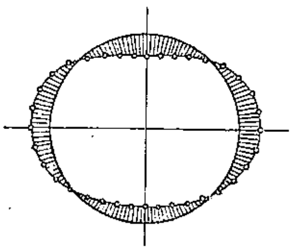
Table IV.

ϕ	nL_2-L_1			
	$n=1$	$n=2$	$n=3$	$n=4$
0°	.0247	.2369	.4491	.6613
10°	.0244	.2636	.5028	.7420
20°	.0255	.3419	.6583	.9747
30°	.0322	.4660	.8998	1.3336
40°	.0495	.6253	1.2011	1.7769
50°	.0798	.8030	1.5262	2.2494
60°	.1213	.9774	1.8335	2.6896
70°	.1665	1.1221	2.0777	3.0333
80°	.2024	1.2091	2.2158	3.2225
90°	.2146	1.2146	2.2146	3.2146
100°	.2103	1.1734	2.1365	3.0996
110°	.2061	1.1335	2.0609	2.9883
120°	.2023	1.0962	1.9901	2.8840
130°	.1987	1.0623	1.9259	2.7895
140°	.1956	1.0330	1.8704	2.7078
150°	.1932	1.0094	1.8256	2.6418
160°	.1914	.9920	1.7926	2.5932
170°	.1903	.9813	1.7723	2.5633
180°	.1899	.9777	1.7655	2.5533



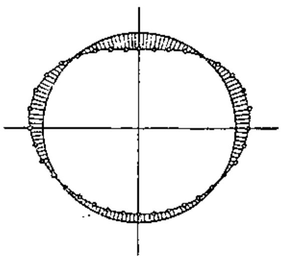
$F_0 = M_1/r^2(g-\frac{\gamma r}{2})$

Fig. 18



$F_0' = M_1'/r^2(\frac{\gamma r}{2}-g)$.

Fig. 19



$F_2 = M_3/w\,r^2$

Fig. 20

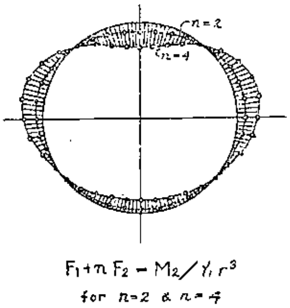


Fig. 21

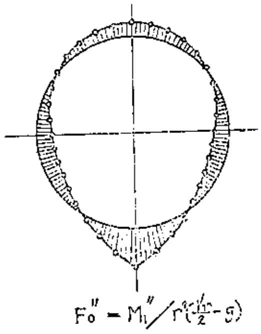


Fig. 22

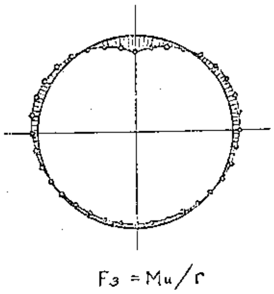


Fig. 23

8. Example

It is required to check the strength of a circular tube section as shown in the **Fig. 24** to be built under water 20 metre deep and with the earth cover 6 metre thick, for the following four cases.

1. Cast iron ring without inside lining, for water pressure, dead and earth load.

2. Cast iron ring with inside concrete lining, for water pressure, dead and earth load.

3. Cast iron ring with inside concrete lining, for water pressure, dead and earth load and also live load of 6 tons per lineal metre.

4. Cast iron ring with inside concrete lining, for water pressure, dead and earth load, and superimposed uniform load of 10 tons per square metre.

Metre and ton are taken as unit throughout the calculation.

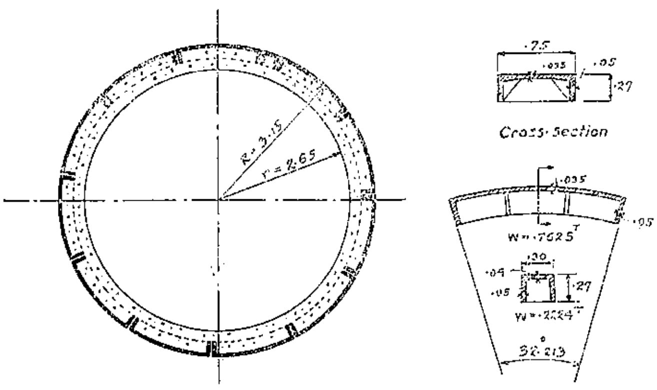


Fig. 24

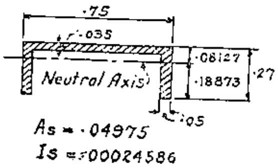


Fig. 25

Case 1.

Depth of water = 20 m

Depth of earth cover = 6 m

$c = 20 + 6 = 26$ m

$r = 3.15 - .08 = 3.07$

Let

$$\gamma = 1\ \text{t/m}^3,\ \gamma' = 2\ \text{t/m}^3,\ \gamma_1 = 1\ \text{t/m}^3.$$

Weight of the tube for .75 m length

$$11 \times .7625 = 8.3875$$
$$1 \times .2224 = .2224$$
$$\text{Bolts \& nuts} = .3821$$
$$8.9920$$

Total weight for 1 m width $= 8.992 \times \frac{4}{3} = 12.0$ t

$$g = \frac{12}{\pi \times 6.14} = .62$$

$$\frac{\gamma r}{2} = 1.535 > g = .62$$

Hence, moment and thrust are to be determined by the equations (33), (34), (45), and (48).

We have

$$r^2\left(\frac{\gamma r}{2} - g\right) = 8.624$$

$$\gamma r = 3.07$$

$$r\left(\frac{\gamma r}{2} - g\right) = 2.809$$

$$\frac{\pi}{\gamma_1}\left(\frac{\gamma r}{2} - g\right) = 2.875$$

$$\gamma_1 r^3 = 28.934$$

$$\gamma_1 r^2 = 9.425$$

$$n = \frac{(6 - 2.875) + 3.07}{3.07} = 2.018$$

By the aid of the tables of coefficients, we get at once, moment and thrust for the points on the neutral axis, subtending central angles at every ten degrees, which are given in the **Table V**.

Maximum positive moment is produced at the crown point, and the maximum negative moment at the point $\phi=80°$.

Now, for unit length of the tube, we have

$$A_s=\frac{4}{3}\times .04975=.06633$$

$$I_s=\frac{4}{3}\times .00024586=.000328$$

Hence we get

For the point $\phi=0°$

$M=8.8257$ tm, $\qquad N=87.273$ t.

$$f=\frac{87.273}{.06633}-\frac{8.8257}{.000328}\times .18873=-3762.6 \text{ t/m}^2 \text{ (tension)}$$

or

$$f=\frac{87.273}{.06633}+\frac{8.8257}{.000328}\times .08127=3502.5 \text{ t/m}^2 \text{ (compression)}$$

For the point $\phi=80°$

$M=-8.2159$ tm, $\qquad N=104.76$ t

$$f=\frac{104.76}{.06633}+\frac{8.2159}{.000328}\times .18873=6306.8 \text{ t/m}^2 \text{ (compression)}$$

or

$$f=\frac{104.76}{.06633}-\frac{8.2159}{.000328}\times .08127=-456.3 \text{ t/m}^2 \text{ (tension)}$$

Case 2.

For unit length of the tube,

$$A_s=\frac{4}{3}\times .04975=.06633$$

$$A_c=\frac{4}{3}\times .32525=.43367$$

$$I_s=\frac{4}{3}\times .00068498=.00091331$$

$$I_c=\frac{4}{3}\times .00922452=.0123$$

$$A_c+mA_s=.43367+7\times .06633=.89793$$

$$I_c+mI_s=.0123+7\times .00091331=.018693$$

$$r=3.15-.175=2.975$$

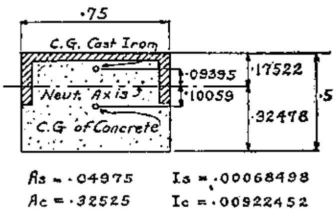


Fig. 26

Weight of cast iron ring 1 m long $=12.000$

" " concrete lining " " $=17.621$

29.621

$$g=\frac{29.621}{\pi\times 5.95}=1.585$$

$$g=1.585>\frac{\gamma r}{2}=1.488$$

Moment and thrust are to be determined by the equations (23), (24), (45), and (48).

We have

$$r^2\left(g-\frac{\gamma r}{2}\right)=.8585 \qquad r\gamma=2.975$$

$$r\left(g-\frac{\gamma r}{2}\right)=.2886 \qquad r^2\gamma_1=8.8506$$

$$c=26 \qquad r^3\gamma_1=26.331$$

$$n=\frac{2.975+6}{2.975}=3.017$$

Hence, by the aid of the tables of coefficients, we get **Table VI.**

For the point $\phi=0°$

$$M=8.9186 \text{ tm} \qquad N=85.927 \text{ t}$$

$$f_c=\frac{N}{A_c+mA_s}-\frac{My}{I_c+mI_s}$$

$$=\frac{85.927}{.89798}-\frac{8.9186}{.018693}\times .32478=-59.27 \text{ t/m}^2 \text{ (tension)}$$

$$f_s=\left\{\frac{85.927}{.89798}+\frac{8.9186}{.018693}\times .17522\right\}\times 7=1255.02 \text{ t/m}^2 \text{ (compression)}$$

If tensile resistance of concrete is neglected,

$$\frac{8.9186}{85.927}=.10379$$

Maximum compression in cast iron ring is

$$7\times\frac{85.927\times 2}{(.17522-.10379)\times 3}=5613.7 \text{ t/m}^2$$

For the point $\phi=80°$

$$M=-8.2843 \text{ tm} \qquad N=105.617 \text{ t}$$

$$f_c=\frac{105.617}{.89798}+\frac{8.2843}{.018693}\times .32478=261.55 \text{ t/m}^2 \text{ (compression)}$$

$$f_s=\left\{\frac{105.617}{.89798}-\frac{8.2843}{.018693}\times .17522\right\}\times 7=279.74\ \text{t/m}^2\ \text{(compression)}$$

Case 3.

In addition to the loading of the Case 2, tube is loaded with a live load, 6 t. per lin. metre, as shown in the figure.

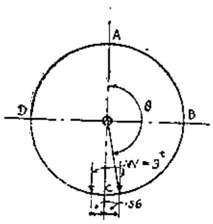


Fig. 27

$$\sin(\pi-\theta)=\sin\theta=\frac{.56}{2.975}=.18824$$

$$\therefore\quad \theta=169°9'=2.95223$$

In this case, we have to find moment and thrust due to live load by the Eqs. (63), (64), (65), (66), (67) and (68) by giving the above value of θ, and add them to the corresponding values in the Case 2.

The result is given in the **Table VII.**

For the point $\phi=0°$

$M=7.7493$ tm $\qquad N=86.848$ t

$$f_c=\frac{86.848}{.89798}-\frac{7.7493}{.018693}\times .32478=-37.93\ \text{t/m}^2\ \text{(tension)}$$

$$f_s=\left\{\frac{86.848}{.89798}+\frac{7.7493}{.018693}\times .17522\right\}\times 7=1185.48\ \text{t/m}^2\ \text{(compression)}$$

If tensile resistance of concrete is neglected, maximum compression in cast iron ring will be

$$7\times\frac{86.848\times 2}{\left\{.17522-\dfrac{7.7493}{86.848}\right\}\times 3}=4713.2\ \text{t/m}^2$$

For the point $\phi=80°$

$M=-7.1891$ tm $\qquad N=105.777$ t

$$f_c=\frac{105.777}{.89798}+\frac{7.1891}{.018693}\times .32478=242.71\ \text{t/m}^2\ \text{(compression)}$$

$$f_s=\left\{\frac{105.777}{.89798}-\frac{7.1891}{.018693}\times .17522\right\}\times 7=352.87\ \text{t/m}^2\ \text{(compression)}$$

Case 4.

In this case, the tube is acted by the superimposed load of 10 t. per square metre in addition to the loading of the Case 2.

Moment and thrust due to the superimposed loading are given by the equations (55) and (56), and since we have $wr^2=88.506$ and $wr=29.75$, M_3 and N_3 are calculated by the aid of the tables of coefficients. The result is added to the corresponding values in the Case 2, and is given in the **Table VIII.**

For the point $\phi=0°$

$M=23.3274$ tm $\qquad N=92.240$ t

$$f_c=\frac{92.240}{.89798}-\frac{23.3274}{.018693}\times .32478=-302.58 \text{ t/m}^2 \text{ (tension)}$$

$$f_s=\left\{\frac{92.240}{.89798}+\frac{23.3274}{.018693}\times .17522\right\}\times 7=2249.66 \text{ t/m}^2 \text{ (compression)}$$

For the point $\phi=80°$

$M=-21.277$ tm $\qquad N=135.566$ t

$$f_c=\frac{135.566}{.89798}+\frac{21.277}{.018693}\times .32478=520.65 \text{ t/m}^2 \text{ (compression)}$$

$$f_s=\left\{\frac{135.566}{.89798}-\frac{21.277}{.018693}\times .17522\right\}\times 7=-339.20 \text{ t/m}^2 \text{ (tension)}$$

It will be seen from the foregoing calculations, that the assumed section is all right for the loadings of the Cases 1, 2 and 3, but stresses are too great for the loading of the Case 4.

Table V. (Case 1)

ϕ	M_1'	M_2	$M_1'+M_2$	N_1'	N_2	$N_1'+N_2$
0°	3.7999	5.0258	8.8257	85.002	2.271	87.273
10°	3.5436	4.6095	8.1531	85.245	2.526	87.771
20°	2.8114	3.7064	6.5178	85.954	3.280	89.234
30°	1.7084	2.2019	3.9103	87.041	3.525	90.566
40°	.3924	.3559	.7483	88.379	5.994	94.373
50°	−.9512	−1.5682	−2.5194	89.816	7.691	97.507
60°	−2.1319	−3.2724	−5.4043	91.194	9.359	100.553
70°	−2.9891	−4.4500	−7.4391	92.361	10.735	103.096
80°	−3.4186	−4.7973	−8.2159	93.205	11.555	104.760
90°	−3.3866	−4.1231	−7.5097	93.656	11.620	105.276
100°	−2.9460	−2.9397	−5.8857	93.844	11.225	105.069
110°	−2.1991	−1.7360	−3.9351	93.921	10.839	104.760
120°	−1.2548	−.6394	−1.8942	93.914	10.481	104.395
130°	−.2251	.3559	.1308	93.853	10.160	104.013

ϕ	M_1'	M_2	$M_1'+M_2$	N_1'	N_2	$N_1'+N_2$
140°	.7839	1.2181	2.0020	93.727	9.877	103.604
150°	1.6782	1.9125	3.5907	93.656	9.651	103.307
160°	2.3768	2.4218	4.7986	93.567	9.484	103.051
170°	2.8209	2.7372	5.5581	93.512	9.387	102.899
180°	3.9736	2.8442	5.8178	93.488	9.350	102.838

Table VI. (Case 2)

ϕ	M_1	M_2	M_1+M_2	N_1	N_2	N_1+N_2
0°	.0608	8.8578	8.9186	81.920	4.007	85.927
10°	.0543	8.2337	8.2880	81.993	4.485	86.478
20°	.0360	6.4537	6.4897	82.213	5.874	88.087
30°	.0086	3.7443	3.7529	82.649	8.029	90.678
40°	−.0233	.4766	.4533	83.051	10.717	93.768
50°	−.0531	−2.8675	−2.9206	83.642	13.617	97.259
60°	−.0739	−5.7744	−5.8483	84.322	16.357	100.679
70°	−.0775	−7.7124	−7.7899	85.068	18.532	103.600
80°	−.0555	−8.2288	−8.2843	85.854	19.763	105.617
90°	0	−7.0383	−7.0383	86.654	19.751	106.405
100°	.0555	−4.9660	−4.9105	87.453	19.054	106.507
110°	.0775	−2.9596	−2.8821	88.240	18.380	106.620
120°	.0739	−1.0796	−1.0057	88.986	17.748	106.734
130°	.0531	.6240	.6771	89.666	17.175	106.841
140°	.0233	2.0959	2.1192	90.257	16.680	106.937
150°	−.0086	3.2861	3.2775	90.739	16.281	107.020
160°	−.0360	4.1629	4.1269	91.095	15.986	107.081
170°	−.0543	4.7001	4.6458	91.314	15.805	107.119
180°	−.0608	4.8818	4.8210	91.388	15.744	107.132

Table VII. (Case 3)

ϕ	M_1+M_2	M_5	$M_1+M_2+M_5$	N_1+N_2	N_5	$N_1+N_2+N_5$
0°	8.9186	−1.1693	7.7493	85.927	.921	86.848
10°	8.2880	−1.1276	7.1604	86.478	.907	87.385
20°	6.4897	−1.0040	5.4857	88.087	.866	88.953
30°	3.7529	− .8020	2.9509	90.678	.798	91.476
40°	.4533	− .5281	− .0748	93.768	.706	94.474
50°	−2.9206	− .1903	−3.1109	97.259	.592	97.851
60°	−5.8483	.2009	−5.6474	100.679	.461	101.140
70°	−7.7899	.6338	−7.1561	103.600	.315	103.915
80°	−8.2843	1.0952	−7.1891	105.617	.160	105.777
90°	−7.0383	1.5710	−5.4673	106.405	0	106.405
100°	−4.9105	1.9114	−2.9991	106.507	−.114	106.393

ϕ	M_1+M_2	M_3	$M_1+M_2+M_3$	N_1+N_2	N_3	$N_1+N_2+N_3$
110°	−2.8821	1.9699	− .9122	106.620	−.134	106.486
120°	−1.0057	1.7455	.7398	106.734	−.059	106.675
130°	.6771	1.2443	1.9214	106.841	.110	106.951
140°	2.1192	.4821	2.6013	106.937	.366	107.303
150°	3.2775	− .5182	2.7593	107.020	.702	107.722
160°	4.1269	−1.7264	2.4005	107.081	1.108	108.189
170°	4.6458	−2.9753	1.6705	107.119	2.093	109.212
180°	4.8210	−2.9335	1.8875	107.132	2.079	109.211

Table VIII. (Case 4)

ϕ	M_1+M_2	M_3	$M_1+M_2+M_3$	N_1+N_2	N_3	$N_1+N_2+N_3$
0°	8.9186	14.4088	23.3274	85.927	6.313	92.240
10°	8.2880	13.3556	21.6436	86.478	7.116	93.594
20°	6.4897	10.3640	16.8537	88.087	9.413	97.500
30°	3.7529	5.8591	9.6120	90.678	12.906	103.584
40°	.4533	.5133	.9666	93.768	17.130	110.898
50°	−2.9206	−4.8501	−7.7707	97.259	21.515	118.774
60°	−5.8483	−9.3905	−15.2388	100.679	25.469	126.148
70°	−7.7899	−12.3112	−20.1011	103.600	28.429	132.029
80°	−8.2843	−12.9927	−21.2770	105.617	29.949	135.566
90°	−7.0383	−11.0632	−18.1015	106.405	29.750	136.155
100°	−4.9105	−7.7974	−12.7079	106.507	28.652	135.159
110°	−2.8821	−4.6058	−7.4879	106.620	27.590	134.210
120°	−1.0057	−1.6728	−2.6785	106.734	26.594	133.328
130°	.6771	1.0090	1.6861	106.841	25.692	132.533
140°	2.1192	3.3278	5.4470	106.937	24.913	131.850
150°	3.2775	5.2042	8.4817	107.020	24.282	131.302
160°	4.1269	6.5848	10.7117	107.081	23.818	130.899
170°	4.6458	7.4345	12.0803	107.119	23.532	130.651
180°	4.8210	7.7177	12.5387	107.132	23.437	130.569

APPLICATION OF GENERAL EQUATIONS TO THE RECTANGULAR TUNNELS WITH CIRCULAR BORE

Subaqueous tunnels built by the trench method is usually designed in such a form. In this method, a circular tube of steel plate is first sunk into the excavated trench, and the entire section is enveloped with the tremie concrete, and after pumping water out of the steel tube another concrete lining is given inside of the steel tube; thus, in the tunnel section built by this method steel tube remains permanently imbedded in the concrete.

In the following analysis, however, for the sake of simplicity, strength of the steel tube will not be taken into consideration, and the section is assumed to consist entirely of mass concrete.

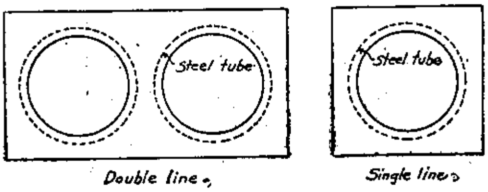


Fig. 28

Tunnels of this kind, when designed for double line, consist of twin tubes as shown in the figure. In this case, stresses in the center wall is determined from the loading that will be applied betweem two crown points of the twin tubes, and the stresses in the outside half section are same as in the case of the single tube, and therefore it will be sufficient to give analysis for the case of a single tube tunnel.

1. Section with Same Thickness at Side and Crown.

As stated before, it is necessary to locate the neutral axis of the section that the general formula derived before can be applied, and this is done in the following way.

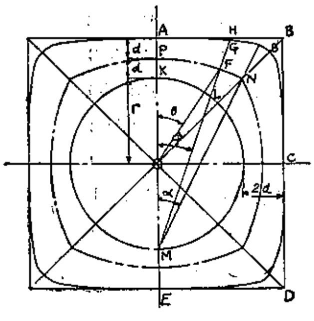


Fig. 29

Let

$ABCDE$=half tunnel section,

$OK=r=$ radius of circular bore,

$AP=PK=d$, $MP=2r+d$.

Draw an arc PN with M as the center and PM as its radius intersecting OB at N.

Draw a straight line $MLFH$ from the point M making an angle ϕ with the vertical axis and intersecting arc PN at F.

Lay off $FG=FL$

Then we have

$$FH=AM\sec\phi-FM=2(r+d)\sec\phi-(2r+d)$$
$$=\{2r(1-\cos\phi)+d+d(1-\cos\phi)\}\sec\phi$$
$$FG=FL=2r(1-\cos\phi)+d$$
$$\therefore\ FH>FG$$

Therefore point G is under the line AB, and the locus of the point G is a curve touching AB at the point A.

If we neglect a small part of concrete $ABB'A$ in the calculation of stresses, and assume AGB' as the exterior line of the tunnel section, arc PFN will become, by construction, the neutral axis of the section. In the triangle OFM,

$$(2r+d):\ r = \sin\theta:\ \sin(\theta-\phi)$$

$$\therefore \qquad \cot\theta = \cot\phi - \frac{r}{2r+d}\operatorname{cosec}\phi$$

For $\theta = 45°$, $\phi = \angle PMN = \alpha$

$$\sin 2\alpha = 1 - \left(\frac{r}{2r+d}\right)^2$$

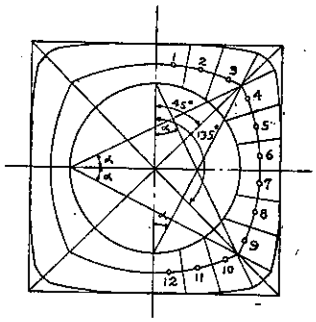


Fig. 30

By dividing the angle α into n equal parts, the half section will be divided into $4n$ parts whose length along the neutral axis are equal, or

$\Delta s =$ constant.

Designate center points of these divisions 1, 2, 3, etc. as shown in the **Fig. 30.**

Since the thickness of the section at these points is given by $4r(1-\cos\phi)+2d$, we can easily find moment of inertia of the section at these points; moment of external forces taken at these points are also calculated for the given loadings; and then these values put in the equation (9) will determine moment and thrust (M_0 and H_0) at the crown point.

Moment and thrust at any point on the neutral axis are to be determined by the equation (6).

In order to show the process clearly, a numerical example will be given in the next article.

2. Example

It is required to check the strength of a tunnel section, as shown in the **Fig. 31,** with the top level under the water 17 metre deep, for the following load.

1. Dead load, and water pressure.
2. Dead load, water pressure, and superimposed uniform load of 10 tons per square metre.

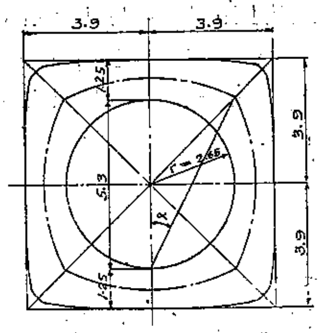


Fig. 31

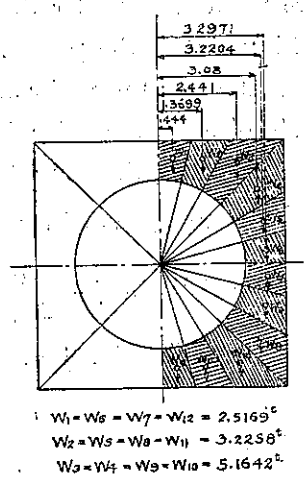


Fig. 32

Case 1

We have

$$r=2.65$$

$$d=\frac{1.25}{2}=.625$$

$$\sin 2\alpha=1-\left(\frac{r}{2r+d}\right)^2=0.8$$

$$2\alpha = 53^\circ\ 7'\ 48''$$

$$\alpha = 26^\circ\ 53'\ 54''$$

$$\frac{\alpha}{6} = 4^\circ\ 25'\ 39''$$

Now, half section is divided into 12 equal parts along the neutral axis, and their center points are designated 1, 2, 3, etc. as shown in the **Fig. 33.**

Co-ordinates, angle ϕ, moment of inertia of the section at these points and also at the points making central angles of 0°, 45°, 135°, and 180° with vertical axis are given in the **Table IX.**

For the calculation of dead load, the half section is divided into 12 parts, for convenience, each part subtending central angle of 15 degrees as shown in the figure, and the concrete being assumed to weigh 2.25 ton per cubic metre, the values of W_1 to W_{12} and their distances of c. g. from the vertical axis are calculated, and shown in the **Fig. 32.**

For water pressure, its intensity at the top of the section is 17 t. per square metre, gradually increasing with the depth, to 24.8 ton per square metre at the bottom level, as shown in the **Fig. 33.**

Total reaction at the bottom of the section is the sum of the vertical water pressure on top level and the weight of the section, and its intensity will be

$$s = 17 + \frac{W_1 + W_2 + \ldots\ldots W_{12}}{3.9} = 28.187 \text{ t/m}^2$$

In figuring moment of external forces at the center pts. on the neutral axis, intersection points of the normals at these center points with the exterior line are shown in the **Fig. 33,** and the moment of all forces acting between this and crown points are calculated and summed up, and are given in the **Table X.**

Moment and thrust at the crown point are now determined by the equation (9) as shown in the **Table X.**

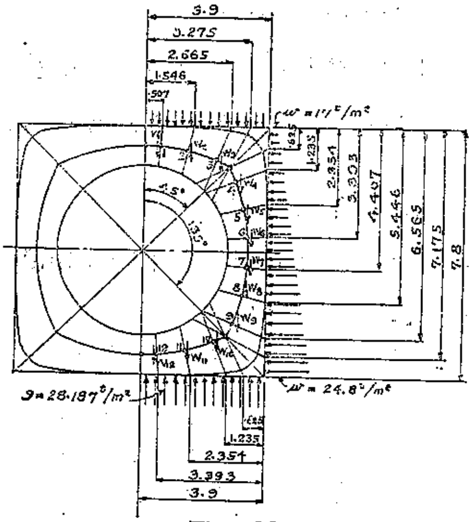


Fig. 33

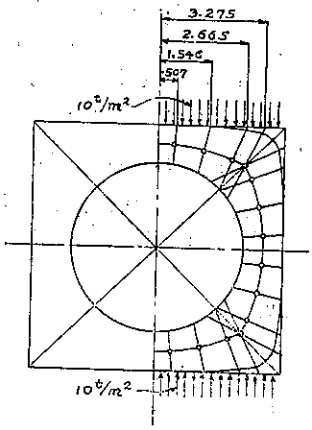


Fig. 34

Moment at center points of all divisions and also at the points making central angles 0°, 45°, 135°, and 180° with the vertical axis are given in the **Table XI.**

Thrust at the same points are given in the **Table XII.**

Diagrams of moments, plotted on the normals and showing their nature at a glance, are given in **Fig. 35, 36** and **37.**

Maximum stress due to dead load and water pressure.

At the crown point,

$M_1 = 17.354$ tm, $N_1 = 69.767$ t

$$f = \frac{69.767}{1.25} \pm \frac{17.354}{.16278} \times .625 = 122.45 \text{ t/m}^2 \text{ or } -10.822 \text{ t/m}^2$$

If tensile resistance of concrete is neglected,

$$f=\frac{2\times 69.767}{\left(\frac{1.25}{2}-\frac{17.354}{69.767}\right)\times 3}=123.6\ \mathrm{t/m^2}$$

Case 2.

In this case, we have to calculate moment and thrust due to superimposed load of 10 ton per square metre and combine with Case 1.

Loading and its reaction are shown in **Fig. 34.**

Moment and thrust at crown point are calculated in the **Table XIII.**

Moment and thrust at other points are given in the **Tables XIV** and **XV.**

Stresses at crown point;

$$M_1+M_2=44.22\ \mathrm{tm},\qquad N_1+N_2=69.767\ \mathrm{t}$$

$$f=\frac{69.767}{1.25}\pm\frac{44.220}{.16278}\times .625=+225.59\ \mathrm{t/m^2}\ \text{or}\ -113.97\ t/m^2$$

The section should be reinforced for tension at the inside of crown point.

Stresses at the point 7;

$$M_1+M_2=-26.411\ \mathrm{tm},\qquad N_1+N_2=130.392\ \mathrm{t}$$

$$f=\frac{130.392}{1.2816}\pm\frac{26.411}{.17542}\times .6408=+198.22\ \mathrm{t/m^2}\ \text{or}\ +5.26\ \mathrm{t/m^2}$$

Table IX.

Pt.	x	y	$a-y$	Thickness	I	ϕ
0°	0	3.275	0	1.25	.16278	0° 0′ 0″
1	.4574	3.2573	.0177	1.2816	.17542	4 25 39
2	1.3613	3.1165	.1585	1.5336	.30058	13 16 57
3	2.2327	2.8382	.4368	2.0314	.69856	22 8 15
45°(A)	2.6497	2.6497	.6254	2.3690	1.10790	26 33 54
45°(B)	〃	〃	〃	〃	〃	63 26 6
4	2.8382	2.2327	1.0423	2.0314	.69856	67 51 45
5	3.1165	1.3613	1.9137	1.5336	.30058	76 43 3
6	3.2573	.4574	2.8176	1.2816	.17542	85 34 21
7	3.2573	−.4574	3.7324	1.2816	.17542	94 25 39
8	3.1165	−1.3613	4.6363	1.5336	.30058	103 16 57
9	2.8382	−2.2327	5.5077	2.0314	.69856	112 8 15
135°(A)	2.6497	−2.6497	5.9247	2.3690	1.10790	116 33 54
135°(B)	〃	〃	〃	〃	〃	153 26 6
10	2.2327	−2.8382	6.1132	2.0314	.69856	157 51 45
11	1.3613	−3.1165	6.3915	1.5336	.30058	166 43 3
12	.4574	−3.2573	6.5323	1.2816	.17542	175 34 21
180°	0	−3.275	6.55	1.25	.16278	180 0 0

Table X.

Pt.	$\frac{1}{I}$	$\frac{a-y}{I}$	$\frac{(a-y)^2}{I}$	m	$\frac{m}{I}$	$\frac{m(a-y)}{I}$
1	5.701	.101	.002	−1.791	−10.211	−.181
2	3.327	.527	.084	−17.771	−59.124	−9.365
3	1.431	.625	.273	−48.069	−68.787	−30.043
4	1.431	1.492	1.555	−94.386	−135.066	−140.824
5	3.327	6.367	12.184	−150.654	−501.226	−959.214
6	5.701	16.062	45.256	−212.506	−1 211.500	−3 413.270
7	5.701	21.277	79.413	−280.044	−1 596.530	−5 958.500
8	3.327	15.425	71.513	−351.014	−1 167.720	−5 414.390
9	1.431	7.884	43.425	−420.364	−601.541	−3 314.150
10	1.431	8.751	53.497	−463.076	−662.662	−4 052.380
11	3.327	21.264	135.908	−470.954	−1 566.860	−10 014.370
12	5.701	37.230	243.252	−474.250	−2 703.700	−17 656.330
Σ	41.836	137.005	696.362		−10 284.927	−50 963.017

By equation (9),

$$M_0=\frac{137.005\times(-50963.017)-(-10284.927)\times 696.362}{41.836\times 696.362-(137.005)^2}=17.354 \text{ tm}$$

$$H_0=\frac{137.005\times(-10284.927)-41.836\times(-50963.017)}{41.836\times 696.362-(137.005)^2}=69.767 \text{ t}$$

Table XI.

Pt.	$a-y$	M_0	$H_0(a-y)$	m	M_1
0°	0	17.354	0	0	17.354
1	.0177	17.354	1.235	−1.791	16.798
2	.1585	17.354	11.058	−17.771	10.641
3	.4368	17.354	30.474	−48.069	−.241
45°(A)	.6254	17.354	43.632	−67.112	−6.126
45°(B)	.6254	17.354	43.632	−67.275	−6.289
4	1.0423	17.354	72.718	−94.386	−4.314
5	1.9137	17.354	133.513	−150.654	.213
6	2.8176	17.354	196.575	−212.506	1.423
7	3.7324	17.354	260.398	−280.044	−2.292
8	4.6363	17.354	323.461	−351.014	−10.199
9	5.5077	17.354	384.256	−420.364	−18.754
135°(A)	5.9247	17.354	413.349	−452.435	−21.732
135°(B)	5.9247	17.354	413.349	−454.692	−23.989
10	6.1132	17.354	426.500	−463.076	−19.218
11	6.3915	17.354	445.916	−470.954	−7.684
12	6.5323	17.354	455.739	−474.250	−1.151
180°	6.55	17.354	453.486	−473.949	−3.109

Table XII.

Pt.	ϕ	$H_0 \cos\phi$	$\Sigma P \sin(\phi-\psi)$	N_1
0°	0° 0′ 0″	69.767	0	69.767
1	4 25 39	69.559	.860	70.419
2	13 16 57	67.901	6.617	74.518
3	22 8 15	64.624	19.236	83.860
45° (A)	26 33 54	62.402	29.776	92.178
45° (B)	63 26 6	31.201	64.216	95.417
4	67 51 45	26.290	63.316	89.606
5	76 43 3	16.029	70.336	86.365
6	85 34 21	5.386	80.445	85.832
7	94 25 39	− 5.386	96.894	91.508
8	103 16 57	−16.029	116.024	99.995
9	112 8 15	−26.290	141.899	115.609
135° (A)	116 33 54	−31.201	154.627	123.426
135° (B)	153 26 6	−62.402	182.215	119.813
10	157 51 45	−64.624	177.145	112.521
11	166 43 3	−67.901	168.092	100.191
12	175 34 21	−69.559	163.441	93.882
180°	180 0 0	−69.767	163.020	93.253

Table XIII.

Pt.	$\frac{1}{I}$	$\frac{a-y}{I}$	$\frac{(a-y)^2}{I}$	m	$\frac{m}{I}$	$\frac{m(a-y)}{I}$
1	5.701	.101	.002	− 1.034	− 5.895	− .104
2	3.327	.527	.082	− 9.095	− 30.259	− 4.793
3	1.431	.625	.273	−23.990	− 34.330	− 14.994
4	1.431	1.492	1.555	−34.640	− 49.570	− 51.683
5	3.327	6.367	12.184	−45.494	−151.358	−289.660
6	5.701	16.062	45.256	−50.985	−290.665	−818.921
7	5.701	21.277	79.413	−50.985	−290.665	−1 084.810
8	3.327	15.425	71.513	−45.494	−151.358	−701.745
9	1.431	7.884	43.425	−34.640	− 49.570	−273.102
10	1.431	8.751	53.497	−23.990	− 34.330	−209.937
11	3.327	21.264	135.908	− 9.095	− 30.259	−193.396
12	5.701	37.230	243.252	− 1.034	− 5.895	− 38.496
Σ	41.836	137.005	696.362		−1 124.154	−3 681.641

By equation (9),

$$M_0 = \frac{137.005 \times (-3681.641) - (-1124.154) \times 696.362}{41.836 \times 696.362 - (137.005)^2} = 26.886 \text{ tm}$$

$$H_0 = \frac{137.005 \times (-1124.151) - 41.836 \times (-3681.641)}{41.836 \times 696.362 - (137.005)^2} = \frac{10}{10363} \cong 0$$

Table XIV.

Pt.	M_0	m	M_2	M_1	M_1+M_2
0°	26.866	0	26.866	17.354	44.220
1	26.866	− 1.034	25.832	16.798	42.630
2	26.866	− 9.095	17.771	10.641	28.412
3	26.866	−23.990	2.876	− .241	2.635
45° (A)	26.866	−33.150	− 7.124	− 6.126	−13.250
45° (B)	26.866	−27.288	− .422	− 6.289	− 6.711
4	26.866	−34.640	− 7.774	− 4.314	−12.088
5	26.866	−45.494	−18.628	.213	−18.415
6	26.866	−50.985	−24.119	1.423	−22.696
7	26.866	−50.985	−24.119	− 2.292	−26.411
8	26.866	−45.494	−18.628	−10.199	−28.827
9	26.866	−34.640	− 7.774	−18.754	−26.528
135° (A)	26.866	−27.288	− .422	−21.732	−22.154
135° (B)	26.866	−33.150	− 7.124	−23.989	−31.113
10	26.866	−23.990	2.876	−19.218	−16.342
11	26.866	− 9.095	17.771	− 7.684	10.087
12	26.866	− 1.034	25.832	− 1.157	24.675
180°	26.866	0	26.866	− 3.109	23.757

Table XV.

Pt.	ϕ	N_1	N_2	N_1+N_2
0°	0 0 0	69.767	0	69.767
1	4 25 39	70.419	.039	70.458
2	13 16 57	74.518	3.552	78.070
3	22 8 15	83.860	10.043	93.903
45° (A)	26 33 54	92.178	14.646	106.824
45° (B)	63 26 6	95.417	34.383	129.800
4	67 51 45	89.606	36.125	125.731
5	76 43 3	86.365	37.957	124.322
6	85 34 21	85.831	38.884	124.715
7	94 25 39	91.508	38.884	130.392
8	103 16 57	99.995	37.957	137.952
9	112 8 15	115.609	36.125	151.734
135° (A)	116 33 54	123.426	34.383	157.809
135° (B)	153 26 6	119.813	14.646	134.459
10	157 51 45	112.521	10.043	122.564
11	166 43 3	100.191	3.552	103.743
12	175 34 21	93.882	.039	93.921
180°	180 0 0	93.253	0	93.253

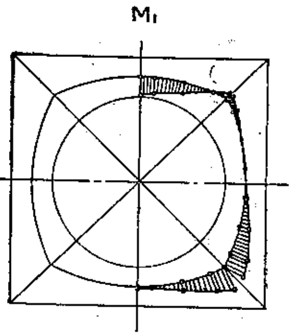


Fig. 35

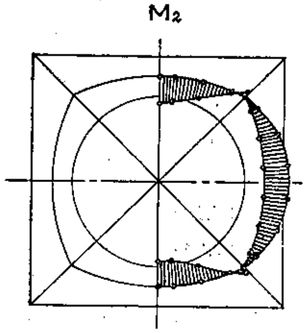


Fig. 36

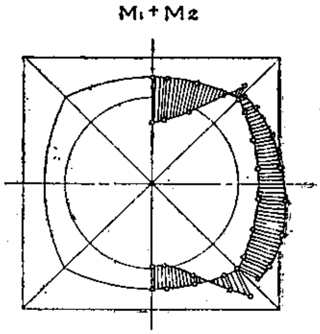


Fig. 37

3. Section with Increased Crown Thickness

Since maximum positive moment is produced at the crown point when the thickness of the section is equal at side and crown, it is advisable to give a slightly greater thickness at the crown point to add strength at its weakest point.

In the **Fig. 38**, let thickness at crown point be $AK=2d'$, Q being its middle point; thickness at side$=2d$, as before.

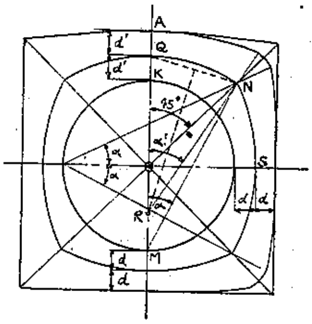


Fig. 38

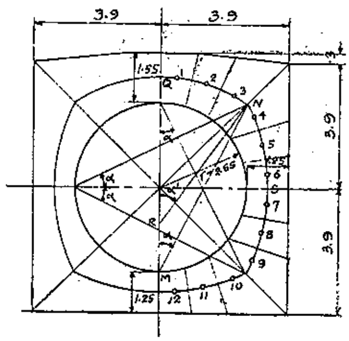


Fig. 39

Let N be the same point on the line subtending central angle of 45 degrees as in the preceeding example.

Join QN, draw perpendicular at the middle point of QN meeting the vertical axis at the point R.

Arc QN, drawn with RQ as radius and R as its center, is taken as the neutral axis for the top part of the section, neutral axis for other parts remaining same as before.

Angle QRN or α' is given by the following equation.

$$\tan\frac{\alpha'}{2}=\frac{(2r+d')-(2r+d)\cos\alpha}{(2r+d)\sin\alpha}$$

Now $\dfrac{MN}{RN}=\dfrac{\sin\alpha'}{\sin\alpha}=\dfrac{\alpha'}{\alpha}$ approximately

$\therefore\quad \alpha.\,MN=\alpha'.\,RN$ or Arc $QN\cong$ Arc NS

Hence by dividing arc QN into n equal parts, we can consider Δs remains practically constant, and the equation (9) can be applied as before, with proper correction for the co-ordinates and moment of inertia for the points on the top part, and also for the change of moments of external forces.

For instance, if we add 0.3 metre to the crown thickness of the preceeding example, we get

$$d'=.775,\quad d=.625,\quad 2r=5.3$$

$$\tan\frac{\alpha'}{2}=\frac{6.075-5.925\times\cos 26°33'54''}{5.925\times\sin 26°33'54''}=.2925$$

$$\therefore\quad \alpha'=32°37'35''$$

$$\frac{\alpha'}{\alpha}\div\frac{\sin\alpha'}{\sin\alpha}=1.0187$$

Thus, difference in arc length is less than 2%.

4. Example

Required to check the strength of the section of the preceeding example when the crown thickness is increased by 0.3 metre.

In the **Fig. 39**, we have

$$\alpha'=32°37'35''$$

$$RQ=RN=MN\frac{\sin\alpha}{\sin\alpha'}=5.925\frac{\sin 26°33'54''}{\sin 32°37'35''}=4.9146$$

Co-ordinates and other dimensions for the points on neutral axis given in the **Table IX** are corrected for increased crown thickness as given in the **Table XVI.**

For moment of external forces, slight increase in the weight of concrete is neglected, but due correction of the increased moment corresponding to the increased lateral water pressure at top is calculated, and the result is given in the **Tables XVII** and **XVIII.**

Moment and thrust at the crown point due to dead load and water pressure are given in the **Table XVII.**

Moment M_1 due to dead load and water pressure at other pts. on the neutral axis is given in the **Table XVIII.**

Moment and thrust at the crown point due to superimposed load of 10 t/m² is given in the **Table XIX.**

Moment M_2 due to the superimposed loading at other points on the neutral axis and also sum of M_1 and M_2 are given in the **Table XX.**

Thrust N due to the dead load and water pressure is given in the **Table XXI.**

Thrust N_2 due to the superimposed loading and also sum of N_1 and N_2 are given in the **Table XXII.**

Moment diagrams, **Figure 40, 41** and **42,** are prepared to show the effect of the increased thickness at the crown point in comparison with the diagrams shown in the preceeding example.

Stresses in the section;

Point $\phi=0°$

$$M_1+M_2=27.688 \text{ tm}$$

$$N_1+N_2=76.585 \text{ t}$$

$$f=\frac{76.585}{1.55}\pm\frac{27.688}{.31032}\times .775$$

$$=118.558 \text{ t/m}^2 \text{ (compression) or } -19.738 \text{ t/m}^2 \text{ (tension)}$$

If tensile resistance in concrete is neglected,

$$f=\frac{2\times 76.585}{\left(\frac{1.55}{2}-\frac{27.688}{76.585}\right)\times 3}=123.47 \text{ t/m}^2 \text{ (compression)}$$

Point $\phi=180°$

$$M_1+M_2=36.512 \text{ tm}$$

$$N_1+N_2=90.490 \text{ t}$$

$$f=\frac{90.490}{1.25}\pm\frac{36.512}{.16278}\times .625$$

$$=212.581 \text{ t/m}^2 \text{ (compression) or } -67.797 \text{ t/m}^2 \text{ (tension)}$$

If tensile resistance in concrete is neglected,

$$f=\frac{2\times 90.490}{\left(\frac{1.25}{2}-\frac{36.512}{90.490}\right)\times 3}=272.36 \text{ t/m}^2 \text{ (compression)}$$

Hence it will be seen from the foregoing calculation, that the section is strong enough if it is subjected to the superimposed load of 10 ton per square metre, beside the dead load and water pressure, and in case due reinforcement is given for tensile stress it will stand safely even for far greater superimposed loading.

Table XVI.

Pt.	x	y	$a-y$	Thickness	I	ϕ
0°	0	3.425	0	1.55	.31032	0° 0′ 0″
1	.4657	3.4025	.0225	1.57	.32249	5 26 16
2	1.3804	3.2267	.1983	1.736	.43598	16 18 48
3	2.2456	2.8815	.5435	2.056	.72425	27 11 19
45° (A)	2.6497	2.6497	.7753	2.2688	.97321	32 37 35
45° (B)	2.6497	2.6497	.7753	2.3690	1.10790	63 26 6
4	2.8382	2.2327	1.1923	2.0314	.69856	67 51 45
5	3.1165	1.3613	2.0637	1.5336	.30058	76 43 3
6	3.2573	.4574	2.9676	1.2816	.17542	85 34 21
7	3.2573	− .4574	3.8824	1.2816	.17542	94 25 39
8	3.1165	−1.3613	4.7863	1.5336	.30058	103 16 57
9	2.8382	−2.2327	5.6577	2.0314	.69856	112 8 15
135° (A)	2.6497	−2.6497	6.0747	2.3690	1.10790	116 33 54
135° (B)	2.6497	−2.6497	6.0747	2.3690	1.10790	153 26 6
10	2.2327	−2.8382	6.2632	2.0314	.69856	157 51 45
11	1.3613	−3.1165	6.5415	1.5336	.30058	166 43 3
12	.4574	−3.2573	6.6823	1.2816	.17542	175 34 21
180°	0	−3.275	6.7000	1.25	.16278	180 0 0

Table XVII.

Pt.	$\frac{1}{I}$	$\frac{(a-y)}{I}$	$\frac{(a-y)^2}{I}$	m	$\frac{m}{I}$	$\frac{m(a-y)}{I}$
1	3.101	.070	.002	−1.791	−5.554	−.125
2	2.294	.455	.090	−19.850	−45.536	−9.032
3	1.381	.750	.408	−53.978	−74.544	−40.483
4	1.431	1.707	2.035	−103.571	−147.869	−176.389
5	3.327	6.866	14.169	−164.247	−546.450	−1 127.720
6	5.701	16.917	50.204	−230.669	−1 315.040	−3 902.230
7	5.701	22.131	85.926	−302.827	−1 726.420	−6 702.190
8	3.327	15.924	76.216	−378.367	−1 258.830	−6 025.120
9	1.431	8.099	45.822	−452.125	−646.990	−3 661.760
10	1.431	8.966	56.155	−497.895	−712.488	−4 464.130
11	3.327	21.763	142.362	−507.183	−1 687.400	−11 037.830
12	5.701	39.888	266.547	−511.187	−2 914.280	−20 390.230
Σ	38.153	143.537	739.936		−11 081.401	−57 537.239

$$M_0=\frac{143.537\times(-57537.239)-(-11081.401)\times 739.936}{38.153\times 739.936-(143.537)^2}=-7.76\text{ tm}$$

$$H_0=\frac{143.537\times(-11081.401)-38.153\times(-57537.239)}{38.153\times 739.936-(143.537)^2}=79.263\text{ t}$$

Table XVIII.

Pt.	$a-y$	M_0	$H_0(a-y)$	m	M_1
0°	0	−7.76	0	0	−7.76
1	.0225	−7.76	1.783	−1.791	−7.768
2	.1983	−7.76	15.718	−19.850	−11.892
3	.5435	−7.76	43.079	−53.978	−18.659
45° (A)	.7753	−7.76	61.453	−74.189	−20.496
45° (B)	.7753	−7.76	61.453	−74.352	−20.659
4	1.1923	−7.76	94.505	−103.571	−16.826
5	2.0637	−7.76	163.575	−164.247	−8.432
6	2.9676	−7.76	235.221	−230.669	−3.208
7	3.8824	−7.76	307.702	−302.827	−2.885
8	4.7863	−7.76	379.353	−378.367	−6.774
9	5.6577	−7.76	448.446	−452.125	−11.439
135° (A)	6.0747	−7.76	481.499	−486.303	−12.564
135° (B)	6.0747	−7.76	481.499	−488.560	−14.821
10	6.2632	−7.76	496.440	−497.895	−9.215
11	6.5415	−7.76	518.499	−507.183	3.556
12	6.6823	−7.76	529.659	−511.187	10.712
180°	6.7000	−7.76	531.062	−510.977	12.325

Table XIX.

Pt.	$\frac{1}{I}$	$\frac{a-y}{I}$	m	$\frac{m}{I}$	$\frac{m(a-y)}{I}$
1	3.101	.070	−1.034	−3.206	−.072
2	2.294	.455	−9.095	−20.864	−4.138
3	1.381	.750	−23.990	−33.130	−17.993
4	1.431	1.707	−34.640	−49.570	−59.131
5	3.327	6.866	−45.494	−151.359	−312.362
6	5.701	16.917	−50.985	−290.670	−862.510
7	5.701	22.132	−50.985	−290.670	−1 128.400
8	3.327	15.924	−45.494	−151.359	−724.446
9	1.431	8.099	−34.640	−49.570	−280.550
10	1.431	8.966	−23.990	−34.330	−215.094
11	3.327	21.763	−9.095	−30.259	−197.935
12	5.701	39.888	−1.034	−5.894	−41.244
Σ	38.153	143.537		−1 110.881	−3 843.875

$$M_0=\frac{143.537\times(-3843.875)-(-1110.882)\times739.936}{38.153\times739.936-(143.537)^2}=35.428\text{ tm}$$

$$H_0=\frac{143.537\times(-1110.882)-38.153\times(-3843.875)}{38.153\times739.936-(143.537)^2}=-1.678\text{ t}$$

Table XX.

Pt.	M_0	$H_0(a-y)$	m	M_2	M_1	M_1+M_2
0°	35.428	0	0	35.428	−7.760	27.668
1	35.428	−.038	−1.034	34.356	−7.768	26.588
2	35.428	−.333	−9.095	26.000	−11.892	14.108
3	35.428	−.912	−23.990	10.526	−18.659	−8.123
45° (A)	35.428	−1.301	−33.150	.977	−20.496	−19.519
45° (B)	35.428	−1.301	−27.288	6.839	−20.659	−13.820
4	35.428	−2.000	−34.640	−1.212	−16.826	−18.038
5	35.428	−3.463	−45.494	−13.529	−8.432	−21.961
6	35.428	−4.979	−50.985	−20.536	−3.208	−23.744
7	35.428	−6.514	−50.985	−22.071	−2.885	−24.956
8	35.428	−8.030	−45.494	−18.096	−6.774	−24.870
9	35.428	−9.493	−34.640	−8.705	−11.439	−20.144
135° (A)	35.428	−10.192	−27.288	−2.052	−12.564	−14.616
135° (B)	35.428	−10.192	−33.150	−7.917	−14.821	−22.738
10	35.428	−10.508	−23.990	.930	−9.215	−8.285
11	35.428	−10.975	−9.095	15.358	3.556	18.914
12	35.428	−11.212	−1.034	23.182	10.712	33.894
180°	35.428	−11.241	0	24.187	12.325	36.512

Table XXI.

Pt.	ϕ	$H_0\cos\phi$	$\Sigma P\sin(\phi-\psi)$	N_1
0°	0° 0′ 0″	78.263	.0	78.263
1	5 26 16	78.906	1.055	79.961
2	16 18 48	76.072	8.089	84.161
3	27 11 19	70.505	23.325	93.830
45° (A)	32 37 35	66.756	31.639	98.395
45° (B)	63 26 6	35.442	61.956	97.398
4	67 51 45	29.869	61.411	91.280
5	76 43 3	18.211	69.175	87.386
6	85 34 21	6.119	80.053	86.172
7	94 25 39	−6.119	97.285	91.166
8	103 16 57	−18.211	117.173	98.962
9	112 8 15	−29.869	143.803	113.934
135° (A)	116 33 54	−35.442	156.888	121.446

Pt.	ϕ	$H_0 \cos\phi$	$\Sigma P \sin(\phi-\psi)$	N_1
135° (B)	153°26 6	−70.895	136.737	115.842
10	157 51 45	−73.420	181.827	108.407
11	166 43 3	−77.143	173.012	95.869
12	175 34 21	−79.026	168.483	89.457
180°	180 0 0	−79.263	168.075	88.812

Table XXII.

Pt.	ϕ	$H_0 \cos\phi$	$\Sigma P \sin(\phi-\psi)$	N_2	N_1	N_1+N_2
0°	0° 0′ 0″	−1.678	0	−1.678	78.263	76.585
1	5 26 16	−1.663	.480	−1.183	79.961	78.778
2	16 18 48	−1.610	4.343	2.733	84.161	86.894
3	27 11 19	−1.493	12.177	10.684	93.830	104.514
45° (A)	32 37 35	−1.413	17.657	16.244	98.395	114.639
45° (B)	63 26 6	−.750	34.883	34.133	97.398	131.531
4	67 51 45	−.631	36.125	35.494	91.280	126.774
5	76 43 3	−.386	37.957	37.571	87.386	124.957
6	85 34 21	−.130	38.884	38.754	86.172	124.926
7	94 25 39	.130	38.884	39.014	91.166	130.180
8	103 16 57	.386	37.957	38.343	98.962	137.305
9	112 8 15	.631	36.125	36.756	113.934	150.690
135° (A)	116 33 54	.750	34.883	35.633	121.446	157.079
135° (B)	153 26 6	1.500	14.646	16.143	115.842	131.988
10	157 51 45	1.551	10.043	11.594	108.407	120.001
11	166 43 3	1.633	3.552	5.185	95.869	101.054
12	175 34 21	1.673	.391	2.064	89.454	91.521
180°	180 0 0	1.678	0	1.678	88.812	90.490

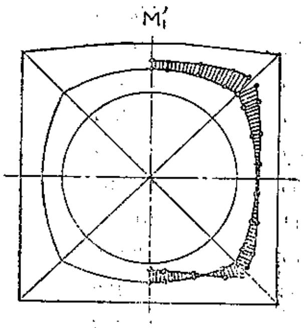


Fig. 40

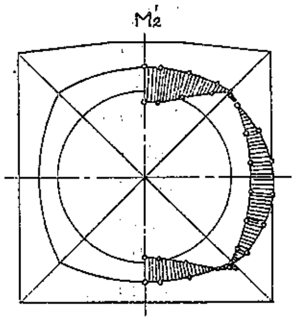


Fig. 41

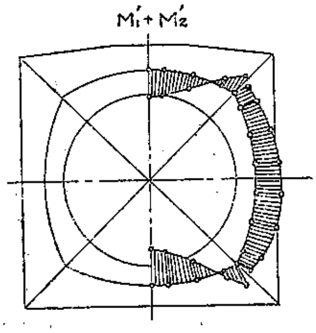


Fig. 42

APPENDIX

Application of Formulas to Some American Existing Tunnels

The formulas presented in this paper are purely theoretical, but since the actual soil condition, owing to the presence of cohesion and friction, differs more or less from the assumption, the result obtained will give only approximate value.

Nevertheless, so long as it is impossible to estimate cohesion or friction even approximately, these formulas, showing general manner of stress distribution, are believed to serve as a guide in determining the tunnel section to be built through the water-bearing soil, by giving proper modification deemed necessary from the knowledge and judgement about the particular soil in the proposed site.

In this connection, it will be interesting to see what result will these formulas give, when applied to the case of a certain American existing tunnel of which careful studies and observations have been made.

North River Tunnel of Pennsylvania R. R., New York

Detailed report of construction of North River Tunnel by Messrs. B. H. M. Hewett and W. L. Brown is to be found in the Vol. LXVIII of the Transactions.

Greater part of soil through which the tunnel has been constructed is stated to be a silt which is very soft acting like a fluid.

It might be questioned, if Hudson River silt behaves like a fluid, whether the Equation (26) may not be applied to find the stresses in the lining; but such a conception will be erroneous from the following reason.

Silt is the water containing in suspension very fine particles of muddy material which are not in a dissolved condition, and so no matter how easily movable or aqueous it may be, water will act, after all, according to the law of fluids, while suspended particles must obey the law of gravitation as any solid material does.

Indeed, the case of the Equation (26) is only imaginary one which will never exist practically, and this is obvious from the fact that the air pressure required in the shield when tunneling through the silty soil was always less than the hydrostatic head above.

Let us now proceed to find the stresses in the cast iron ring before it was lined with concrete.

Foot and pound are taken as unit in the following calculations.

Two kinds of cast iron segments, ordinary and heavy type, was used in North River Tunnel as shown in the **Fig. 44.**

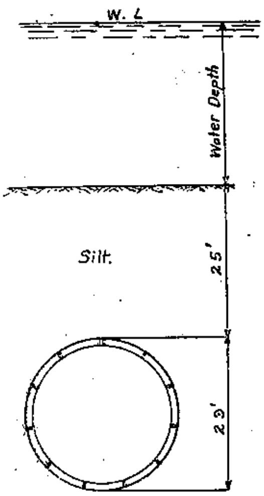


Fig. 43

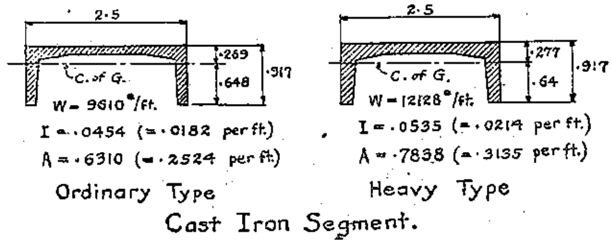


Fig. 44

Greatest water depth is about 40 ft. for the section of ordinary type segment and 50 ft. for heavy type segment. Depth of silt above the crown is taken at 25 ft.

Ordinary type segment.

$$r=\frac{23}{2}-.269=11.231 \text{ ft.}$$

$$g=\frac{9610}{\pi\times 2\times 11.231}=136 \ \#/\text{ft}^2.$$

$$\gamma=64 \ \#/\text{ft}^3. \qquad \gamma'=108 \ \#/\text{ft}^3. \qquad \gamma_1=108-64=44 \ \#/\text{ft}^3.$$

$$\frac{\gamma r}{2}=\frac{11.231\times 64}{2}=360$$

$$\therefore \quad \frac{\gamma r}{2}>g$$

$$\frac{\pi}{\gamma_1}\left(\frac{\gamma r}{2}-g\right)=\frac{3.1416}{44}(360-136)=16 \text{ ft.}$$

$$n=\frac{(25-16)+11.231}{11.231}=1.8$$

$$c=40+25=65 \qquad k=65+\frac{11.231}{2}=70.6$$

Moment and thrust at the crown point:

from (33)

$$M_1'=r^2\left(\frac{\gamma r}{2}-g\right)F_0'=\overline{11.231}^2\times224\times.4406=12449 \text{ ft. lbs.},$$

from (34)

$$N_1'=r\left\{\gamma k+\left(\frac{\gamma r}{2}-g\right)L_0'\right\}=11.231\times(64\times70.6+224\times.1667)=51166 \text{ lbs.},$$

from (45)

$$M_2=\gamma_1 r^3(F_1+nF_2)=44\times\overline{11.231}^3\times(-.1548+1.8\times.1628)=8614 \text{ ft. lbs.},$$

from (48)

$$N_2=\gamma_1 r^2(nL_2-L_1)=44\times\overline{11.231}^2\times(1.8\times.2122-.1875)=1079 \text{ lbs.}$$

$$M_1'+M_2=12449+8614=21063 \text{ ft. lbs.}$$

$$N_1'+N_2=51166+1079=52245 \text{ lbs.}$$

Maximum compression in cast iron segment:

$$f=\frac{52245}{.2524}+\frac{21063}{.0182}\times.269=518320 \ \#/\text{ft}^2. \ (=3600 \ \#/\text{in}^2.)$$

Maximum tension in cast iron segment:

$$f=\frac{52245}{.2524}-\frac{21063}{.0182}\times.648=-542940 \ \#/\text{ft}^2. \ (=-3770 \ \#/\text{in}^2.)$$

If maximum pressure of compressed air used is 40 $\#/\text{in}^2$., intensity of axial tension in the cast iron ring is

$$f=\frac{-40\times144\times11.231}{.2524}=-256300 \ \#/\text{ft}^2. \ (=-1780 \ \#/\text{in}^2.)$$

$$\therefore \ \text{Total tension}=-3770-1780=-5550 \ \#/\text{in}^2.$$

Heavy type segment.

$$r=\frac{23}{2}-.277=11.223$$

$$g=\frac{12128}{\pi\times2\times11.223}=172 \qquad \frac{\gamma r}{2}=\frac{11223\times64}{2}=359$$

$$\therefore \ \frac{\gamma r}{2}>g$$

$$\frac{\pi}{\gamma_1}\left(\frac{\gamma_2}{2}-g\right)=13.35 \text{ ft.}$$

$$n=\frac{(25-13.35)+11.223}{11.223}=2.04$$

$$c=50+25=75$$

$$k=75+11.223/2=80.612$$

By similar process as before, we get

$$M_1'+M_2=21400 \text{ ft. lbs.}$$

$$N_1'+N_2=59610 \text{ lbs.}$$

Maximum compression in cast iron segment :

$$f=\frac{59610}{.3135}+\frac{21400}{.0214}\times.277=467150 \text{ #/ft}^2 \ (=3240 \text{ #/in}^2.)$$

Maximum tension in cast iron segment :

$$f=\frac{59610}{.3135}-\frac{21400}{.0214}\times.64=-449850 \text{ #/ft}^2 \ (=-3120 \text{ #/in}^2.)$$

Intensity of axial tension due to the internal pressure of the compressed air at 40 lbs. per sq. in.

$$f=\frac{-40\times144\times11.223}{.3135}=-206200 \text{ #/ft}^2. \ (=-1430 \text{ #/in}^2)$$

Total tension $=-3120-1430=-4550$ #/in^2.

Besides these stresses cast iron lining is subjected to the longitudinal thrust from shoving shield by hydraulic rams, acting in the normal direction to that of ring stresses.

Several phenomena, which are stated as observed during the period of construction, seem to be explained in the following way.

It is stated that when the shield was pushed in silt with all doors closed, lining will rise immediately after leaving shield, and at the same time increase its vertical diameter. After some time, however, as the soil recovers its normal condition, shortening of the vertical and lengthening of the horizontal diameter will gradually take place.

When the shield was pushed with its doors partially closed and admit a part of muck to flow in, the lining will not rise after leaving the shield, quantity of muck thus taken in varied from nothing to the full volume displaced by the tunnel and averaged 33% of the latter.

Now, pushing the shield blind will press surrounding soil and squeeze or drives out water from silt just like compressed air does, and naturally this water will seek to fill the void around the cast iron lining, simultaneously left behind the shield.

As soon as cast iron lining leaves the shield it is thus surrounded with water or silt of very thin density, and being lighter than the displaced water will float up until it is prevented by the tail of the shield and the weight of the finished part of the tunnel or by the weight of materials which are laid on the invert.

This holding down of the lining from floating up causes the moment which has different sign with that of M_u and will give negative deflection to the crown point; in other words, vertical diameter will increase by holding down the cast iron lining from floating up. This elastic deformation when calculated for the case of ordinary type segments is about 0.17 ins., and much greater elongation observed in North River Tunnel is probably due to the inelastic deformation aggravated by the loosening of joint as the result of internal pressure of the compressed air.

After the disturbance caused by shoving the shield is over, fine materials contained in the silt will gradually settle around the lining and by the weight of overlying silt bending moment M_2 will be produced in the lining; this moment having same sign as M_u causes positive deflection, that is to say, crown point of the lining begins to flatten; the above reasoning seems to explain why shortening of vertical diameter was observed when shoving the shield blind and phenomena of opposite nature followed to it subsequently.

When the shield was shoved with doors partially opened, not only pressure on silt is relieved but also the material admited will act to weigh down the lining from floating up.

If there is no friction and nothing lying on the invert beforehand, about 40 and 35 per cent of the volume displaced by the tunnel will be needed to balance buoyancy of the ordinary and heavy type cast iron lining.

It is stated also that a certain number of cast iron segments, mostly in roof but in some case in the invert, became cracked during the period of construction; these segments were not taken out but was reinforced by turnbuckles put inside by bolting to the flanges of the segments.

This fact will be explained by the formulas given in the Eqs. (33), (45) and (55), for maximum positive moments are produced at the crown and invert points, the former being usually greater, and let us try to figure out approximately how such a stress as to cause breakage of the segment was produced.

Nothing is stated about the point where these accident did happened except one which occured when holing in the silt under the Fowler warehouse building which is on New Jersey side of the river.

From the profile of the tunnel depth of earth above the crown is about 60 ft. at this point. There is no data to know the intensity of loading from the warehouse building, but let us now assume arbitrarily a superimposing load of one ton per sq. ft.

We have

$r=11.231$ ft. $\quad g=136$ #/ft^2.

$\gamma=64$ #/ft^3. $\quad \gamma'=108$ #/ft^3. $\quad \gamma_1=44$ #/ft^3.

$\frac{\gamma r}{2}=360 \quad \frac{\pi}{\gamma_1}\left(\frac{\gamma r}{2}-g\right)=16$ ft. $\quad n=\frac{(60-16)+11.231}{11.231}=4.9$

$c=60 \quad k=65.6$

Moment and thrust at the crown point:

from (33)

$$M_1'=r^2\left(\frac{\gamma r}{2}-g\right)F_0'=\overline{11.231}^2\times 224\times .4406=12449 \text{ ft. lbs.},$$

from (34)

$$N_1'=r\left\{\gamma k+\left(\frac{\gamma r}{2}-g\right)L_0'\right\}=11.231\times(64\times 65.6+224\times .1667)=47571 \text{ lbs.},$$

from (45)

$$M_2=\gamma_1 r^3(F_1+nF_2)=44\times\overline{11.231}^3\times(-.1548+4.9\times .1628)=40073 \text{ ft. lbs.},$$

from (48)

$$N_2=\gamma_1 r^2(nL_2-L_1)=44\times\overline{11.231}^2\times(4.9\times .2122-.1875)=4730 \text{ lbs.},$$

from (55)

$$M_3=wr^2F_2=2240\times\overline{11.231}^2\times .1628=45998 \text{ ft. lbs.},$$

from (56)

$$N_3=wrL_2=2240\times 11.231\times .2122=5339 \text{ lbs.}$$

$$\therefore\ M_1'+M_2+M_3=98520 \text{ ft. lbs.}$$

$$N_1'+N_2+N_3=57640 \text{ lbs.}$$

Maximum compression in cast iron segment:

$$f=\frac{57640}{.2524}+\frac{98520}{.0182}\times .269=1227770 \text{ \#/ft}^2\ (=8520 \text{ \#/in}^2.)$$

Maximum tension in cast iron segment:

$$f=\frac{57640}{.2524}-\frac{98520}{.0182}\times .648=-3279380 \text{ \#/ft}^2.\ (=-22800 \text{ \#/in}^2.)$$

If we add -1330 #/in^2. corresponding internal pressure of the compressed air at 30 #/in^2.

Intensity of total tensile stress is

$$-22800-1330=-24130 \text{ \#/in}^2.$$

This is nearly equal to the ultimate strength for cast iron, and we can imagine that when the load of the warehouse on the ground is more than one ton per sq. ft., it was sufficient to cause the breakage of cast iron segment by tensile stress.

However, when crack is opened once, the segment will not stand bending moment any more, and the ring probably had to read just itself in a certain way so that the web of the segment may act like a hinged joint for transmission of stresses until the turnbuckle reinforcement was put in to resist bending moment.

From the foregoing fact it may be said, that by reinforcing a portion near the crown and invert or by using some special segment of greater strength at these weakest points, a greater strength can be added without materially increasing the cost of cast iron lining.

Negative moment produced at sides causes tension in the outside fiber of the lining, but since the neutral axis is nearer to the outside fiber its intensity is not great, some times less than the compression due to the axial thrust, and for this reason cast iron segments at sides will be subjected to much smaller stresses that at the crown and invert of the lining.

Oakland Estuary Subway, California.

The subway tunnel between Oakland and Alameda, construction work of which is now nearly finished, consist of a reinforced concrete tube, 32 ft. inner diameter and 2 ft. 6 in. thick, with a tie rod $1\frac{5}{8}$ in. diameter placed 8 ft. $5\frac{3}{8}$ in. below the crown point and spaced 12 in. ctrs., as shown in the **Fig. 45.**

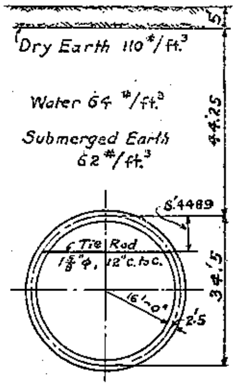


Fig. 45

Load assumption for designing was as follows.

Water pressure considered as acting radially over entire circumference.

Dry earth weighing 110 #/ft³. and submerged earth weighing 62 #/ft³. (weight was reduced for buoyancy) was assumed to act vertically on the upper surface of the tube.

Lateral pressure, taken at 33 % of the vertical earth pressure (excluding weight of water), was assumed to act on the tube horizontally.

Vertical upward reaction was assumed to act concentrated on two points at a distance of 31 ft. and equidistant from the center line of the tunnel.

Analysis was made, under this assumption, by the method of least work and the main result obtained is as follows.

Maximum positive moment near crown point.................45 652 ft. lbs.
Thrust at the same point.................................102 642 lbs.
Maximum positive moment near the working point of
the concentrated reaction................................58 351 ft. lbs.
Thrust at the same point.................................87 997 lbs.
Maximum negative moment at sides.........................−52 486 ft. lbs.
Thrust at the same point.................................129 420 lbs.

Now let us try to see what result will be obtained when the formulas presented in this paper are applied for this case.

Since this case is only different from the case of ordinary tube in using tie rod to connect upper part of the section, our formulas derived for the circular tube with uniform cross-section will be applicable, if we consider the effect of tie rod pull separately and combine it with the effect of other loadings.

Effect of the pull of Tie Rod.

In the **Fig. 46** let P be the force acting on the tie rod fastened to the circular tube.

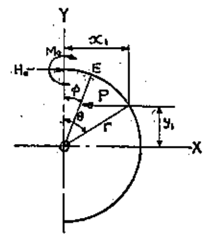


Fig. 46

Moment of the external force P taken at any point E on the neutral axis, subtending central angle ϕ with the vertical axis will be

$m=0$ between $\phi=0$ & $\phi=\theta$,

$m=-P(y_1-y)$ between $\phi=\theta$ & $\phi=\pi$

$$\therefore \quad \int_0^\pi m ds=-Pr\{y_1(\pi-\theta)+r\sin\theta\}$$

$$\int_0^\pi m(a-y)ds=-Pr^2\left\{y_1(\pi-\theta)+(y_1+r)\sin\theta+\frac{\pi r-y_1\sin\theta-\theta r}{2}\right\}$$

Putting these values in (12) we get, as before

$$M_0=-\frac{P}{\pi}(r-y_1)(\pi-\theta-\sin\theta) \qquad \text{(81)}$$

$$H_0=\frac{P}{\pi r}\{y_1\sin\theta+r(\pi-\theta)\} \qquad \text{(82)}$$

Moment (M_t) and thrust (N_t) due to the pull of tie rod at any point on the neutral axis can be expressed in the following form;

for the value of ϕ between 0 & θ,

$$M_t=\frac{P}{\pi}\left[-(r-y_1)(\pi-\theta-\sin\theta)+\{y_1\sin\theta+r(\pi-\theta)\}(1-\cos\phi)\right] \quad \text{..(83)}$$

$$N_t=\frac{P}{\pi r}\{y_1\sin\theta+r(\pi-\theta)\}\cos\phi \quad \text{........(84)}$$

for the value of ϕ between θ & π,

$$M_t=\frac{P}{\pi}\left[-(r-y_1)(\pi-\theta-\sin\theta)+\{y_1\sin\theta+r(\pi-\theta)\}(1-\cos\phi)-\pi(y_1-y)\right] \quad \text{........(85)}$$

$$N_t=P\left[\frac{1}{\pi r}\{y_1\sin\theta+r(\pi-\theta)\}-1\right]\cos\phi \quad \text{........(86)}$$

Putting in these equations following values,

$r=17.25,\qquad y_1=17.25-8.4489=8.8021,$

$\theta=58°19'.1=1.0353,\qquad \sin\theta=.86001,$

we get

for $\phi=0$ to $\phi=\theta$

$M_t=P(10.6235-13.9749\cos\phi)$

$N_t=P\times.81014\cos\phi$

for $\phi=\theta$ to $\phi=\pi$

$M_t=P(1.8214+3.2751\cos\phi)$

$N_t=-P\times.18986\cos\phi$

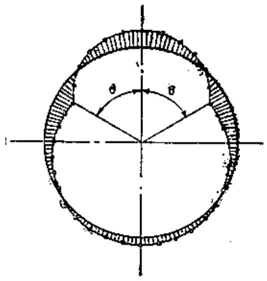


Fig. 47

Fig. 47 shows M_t or effect of tie rod pull on a circular tube.

Now, sectional area of the tie rod is 2.0739 in². per lin. ft. of the tunnel and allowable unit stress for steel rod is 16 000 #/in.² by specification, and therefore we have,

$P=2.0739\times16\,000=33\,180$ lbs.

Substituting this value of P in the above equation, moment and thrust for the value of ϕ at every 10 degrees are calculated as given in the **Tables XXIII** & **XXIV.**

Moment and thrust due to other loadings are obtained from the Eqs, (33), (34), (45), (48), (55), (56) by putting the following values,

$r=17.25,\quad \gamma=64,\quad \gamma_1=62,\quad g=2.5\times150=375,\quad \frac{\gamma r}{2}=552,\quad \frac{\pi}{\gamma_1}\left(\frac{\gamma r}{2}-g\right)=8.96,\quad c=44.25,\quad n=\frac{(44.25-8.96)+17.25}{17.25}=3.05,\quad w=550.$

These moment and thrust are tabulated and combined with M_t as given in the **Tables XXIII & XXIV.**

From the difference in the original assumptions, especially in the lower half circle, where reaction to the vertical forces was assumed to be concentrated in one case and distributed in the other, wider difference in results has been expected.

The results obtained, however, must be considered generally to agree pretty well, and the difference in the maximum moment and thrust near the crown point is about 5 %.

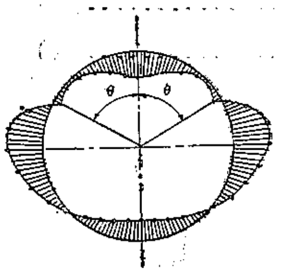


Fig. 48

Fig. 48 is the moment diagram prepared from the result given in the **Table XXIII,** positive moment being plotted inside and negative moment outside of the neutral axis on the normals at its corresponding point, and will show clearly effect of tie rod pull on the moment due to other loadings.

Table XXIII

ϕ	M_1' ft.lbs.	M_2 ft.lbs.	M_3 ft.lbs.	M_t ft.lbs.	Total ft.lbs.
0°	23 206	108 740	26 644	−111 200	47 390
10°	21 642	101 070	24 696	−104 160	43 248
20°	17 170	79 242	19 165	− 83 235	32 340
30°	10 434	45 954	10 834	− 49 077	18 145
40°	2 397	5 824	949	− 2 717	6 453
50°	− 5 810	− 35 229	− 8 969	54 435	4 427
60°	−13 020	− 70 904	−17 364	114 770	13 482
70°	−18 255	− 94 676	−22 765	97 599	−38 097
80°	−20 878	−100 980	−24 025	79 304	−66 579
90°	−20 684	− 86 402	−20 457	60 434	−67 109
100°	−17 992	− 60 943	−14 419	41 565	−51 789
110°	−13 431	− 36 312	− 8 576	23 270	−35 049
120°	− 7 664	− 13 239	− 3 093	6 099	−17 897
130°	− 1 375	7 670	1 866	− 9 417	− 1 256
140°	4 788	25 746	6 154	− 22 811	13 877
150°	10 250	40 321	9 623	− 33 675	26 519
160°	14 516	51 077	12 176	− 41 681	36 088
170°	17 228	57 697	13 748	− 46 581	42 092
180°	18 161	59 702	14 271	− 48 234	43 900

Table XXIV.

ϕ	N_1' lbs.	N_2 lbs.	N_3 lbs.	N_t lbs.	Total lbs.
0°	58 883	8 481	2 013	26 880	96 257
10°	59 216	9 498	2 269	26 472	97 455
20°	61 183	12 466	3 002	25 260	191 911
30°	61 690	17 002	4 116	23 279	106 087
40°	63 585	22 692	5 463	20 592	112 332
50°	65 690	28 826	6 861	17 278	118 655
60°	67 815	34 618	8 122	− 3 150	107 405
70°	69 778	39 215	9 066	− 2 155	115 904
80°	71 433	41 810	9 551	− 1 094	121 700
90°	72 692	41 782	9 488	0	123 962
100°	73 663	40 308	9 137	1 094	124 202
110°	74 484	38 880	8 799	2 155	124 318
120°	75 170	37 542	8 481	3 150	124 343
130°	75 727	36 330	8 193	4 049	124 299
140°	76 173	35 282	7 945	4 826	124 226
150°	76 502	34 435	7 744	5 456	124 137
160°	76 725	33 811	7 596	5 920	124 052
170°	76 871	33 430	7 505	6 204	124 010
180°	76 914	33 300	7 474	6 300	123 988