論 說 報 告

土木學會誌 第十三卷第四號 昭和二年八月

On the Thermal Bending of a Plane Wall Heated on one Surface.

By Noboru Yamaguti, C. E., Member.

Synopsis.

This paper is a study of the bending caused by heating uniformly one side of a wall or a slab. The wall as built is considered that its boundaries are capable of moving freely in its original plane. The thermal bending in such case will be naturally greater in a wall built with materials of low heat conductivity like concrete, etc. The differential equation expressing the deflection due to thermal bending takes the form of a bi-harmonic equation when the wall is considered as a "thin plate." Formulas for the deflections, moments and shears etc. are deduced for the cases of infinitely extended strips, as well as rectangular and circular plates with various boundary conditions. Numerical examples are given with diagrams showing the results of computations.

I. Preliminary.

The structures with large plane surfaces exposed to the sun such as retaining walls, dams, bridges and buildings, are generally built with expansion joints at proper distances. This practice easily rejects the direct stresses due to their temperature change as a whole. But the walls, which are built with materials ofl ow heat conductivity like concrete are not free from another kind of thermal stresses even if they are sufficiently provided with expansion joints. These thermal stresses are due to the non-uniform temperature distributions along the thickness of the wall. These "secondary thermal stresses" might naturally be enormous when one side of a wall is heated by fire of intense heat rising sometimes to as high temperature as more than one thousand degree.

My aim is to compute the above mentioned stresses occurring in plane walls or flat slabs to be treated as "thin plates" in Mechanics. To fix ideas we begin with the thermal flexure of beams. Here we assume that beams yield to freely in longitudinal directions[1] and we also assume that the amount of

(1) When beams are constrained longitudinally at their supports the direct stresses have to be taken into considerations.

deflection is so little that the bending will not deviate from the usual condition of bending called by the name of Bernoulli, i. e., a plane section of a beam remains a plane after the beam being bended.

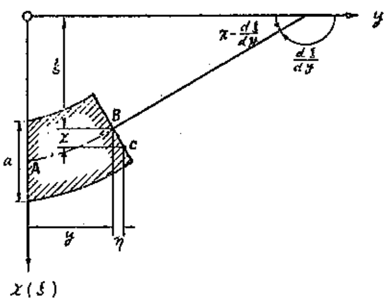


Fig. 1.

Take the co-ordinate axes as are shown in **Fig. 1.** and take the positive direction of x so that stress may be in tension.

Neutral axis AB is either elongated or contracted by the rise and fall of temperature, but not strained by the bending action.

Referring to **Fig. 1.** the relative horizontal displacement of C to B is

$$\eta = x \tan\left(\pi - \frac{d\xi}{dy}\right) = -x \tan \frac{d\xi}{dy} \doteqdot -x\frac{d\xi}{dy}$$

Where ξ means vertical deflection of a beam at B. We know that one part of relative displacement is attributable to the unequal elongations due to difference of temperature at B and C, and that the other part of it to the strain generated by bending stress.

Therefore we may put, as R. Lorenz did in a cylindrical tube(1),

$$\frac{d\eta}{dy} = \epsilon_y + \alpha\,(\theta - \theta_m)$$

here ϵ_y means strain along y axis due to normal stress along y axis generated by bending.

θ_m, θ: temperatures at B and C resp.

α: coefficient of linear expansion.

ϵ_y follows Hooke's law of elasticity. σ_y being normal stress along y axis and E Young's Modulus, which is assumed to be constant independent of temperature(2), we have:—

$$\epsilon_y = \frac{1}{E} \cdot \sigma_y$$

(1) R. Lorenz: Temperaturspannungen in Hohlzylindern: Z. V. D. I. 1907 or H. Lorenz: Technische Elastizitätslehre S. 583.

(2) By Lord Kervin E for steel is as follows: 1 728 000 kg/cm² at 15°C; 2 129 000 ,, 100°C [Encyc. Brit.]

by Dadourian, it is expressed in a following equation:

$$E = E_0\,(1 - .00268\,t - .000000\,71\,t^2) \quad t \text{ in } {}^\circ\mathrm{C}$$ [Phil. Mag. 1921.]

and
$$\frac{d\eta}{dy} = -x\frac{d^2\xi}{dy^2}$$

Putting these values in the above equation, we have:—

$$\frac{1}{E}\sigma_y + \alpha(\theta - \theta_m) = -x\frac{d^2\xi}{dy^2}$$

Multiplying each term by $x\,dx$ and integrating it along the total depth a, we get:—

$$\frac{1}{E}\int_{(a)} \sigma_y x\,dx + \alpha\int_{(a)} (\theta-\theta_m)\,x\,dx = -\frac{d^2\xi}{dy^2}\int_{(a)} x^2\,dx$$

As $\int_{(a)} \sigma_y x\,dx = M$, Bending moment at section B which must be taken to be positive as to give tension on a positive side of x.

$\int_{(a)} x^2\,dx = I$ Moment of inertia of the section about Neutral Axis.

$\alpha\int_{(a)} (\theta-\theta_m)\,x\,dx = \Phi$

The equation becomes:—

$$\frac{1}{E}M + \Phi = -I\frac{d^2\xi}{dy^2}$$

or

$$\left.\begin{aligned} -\frac{d^2\xi}{dy^2} &= \frac{M}{EI} + \frac{\Phi}{I} \\ M &= -EI\frac{d^2\xi}{dy^2} - E\Phi \\ \sigma_y &= -Ex\frac{d^2\xi}{dy^2} - E\alpha(\theta-\theta_m) \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(1)$$

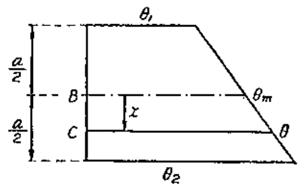


Fig. 2.

If the temperature distribution is uniformly-varrying with respect to x as in **Fig. 2,** Neutral Axis is situated in the middle depth,

$$\theta - \theta_m = \frac{\theta_2-\theta_1}{a}x = \Theta x, \qquad \Theta = \frac{\theta_2-\theta_1}{a},$$

$$\theta = \theta_m + \Theta x, \qquad \Phi = \alpha\Theta I$$

The equation (1) becomes:—

$$\left.\begin{aligned} -\frac{d^2\xi}{dy^2} &= \frac{M}{EI} + \alpha\Theta \\ M &= -EI\frac{d^2\xi}{dy^2} - E\alpha\,\Theta I \\ \sigma_y &= -Ex\frac{d^2\xi}{dy^2} - E\alpha\,\Theta x \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(1)_a$$

Differentiating the first equation of (1) with respect to y, we obtain:—

$$-\frac{d^3\xi}{dy^3}=\frac{V}{EI}+\frac{1}{I}\frac{d\Phi}{dy}, \qquad V: \text{ shear at } B$$

If Φ is constant in any part along the beam, as in the case that the top or bottom face of the beam is heated uniformly, the second term of the right hand side member vanishes, and we have:—

$$-\frac{d^3\xi}{dy^3}=\frac{V}{EI} \quad \dots\dots\dots\dots\dots\dots\dots (2)$$

Here we find temperature term no more. Differentiating this equation again, w. r. t. y,

$$-\frac{d^4\xi}{dy^4}=\frac{p}{EI}, \qquad p: \text{ distributed load at } B.$$

If there is no distributed load, we will have of course,

$$\frac{d^4\xi}{dy^4}=0 \quad \dots\dots\dots\dots\dots\dots\dots\dots (3)$$

For an example, we take a cantilever with a concentrated load P on its free end and heated uniformly all over its bottom face.

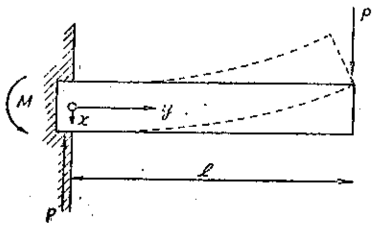


Fig. 3.

Shear at y: $V=-P$

Bending moment at y: $M=-P(l-y)$

Putting these values in the equations (1),

$$-\frac{d^2\xi}{dy^2}=-\frac{P(l-y)}{EI}+\frac{\Phi}{I}$$

By integrating twice with proper boundary conditions, we obtain the following equation of the elastic line,

$$-\xi=-\frac{P}{6EI}(3ly^2-y^3)+\frac{\Phi y^2}{2I}$$

By this equation, we know immediately that the term due to load is cubic w. r. t. y and that the term due to temperature is quadratic. Hence we cannot take off the temperature curvature by any concentrated load.

If we take off P

$$\xi \doteq -\frac{\Phi}{2I}y^2 \quad \dots\dots\dots\dots\dots\dots\dots (4)$$

Cantilever bends into an ordinary parabola or a circular arc for small value of ξ as is shown in the dotted line in **Fig. 3,** and

$$\frac{d^2\xi}{dy^2}=-\frac{\Phi}{I}$$

Therefore we see by equation (1) & $(1)_a$ that M vanishes at any section, but σ_y does not vanish excepting for the case of uniformly varrying distribution of temperature. In the cases of non-linear distributions of temperature the position of Neutral Axis is obtainable from the statical condition that the resultant force of stresses σ_y over all the section be zero i. e.,

$$\int_{(a)}\sigma_y dx=\frac{E\Phi}{I}\int_{(a)} x\,dx-E\alpha\int_{(a)}(\theta-\theta_m)\,dx$$

$$=E\alpha\left\{\frac{G}{I}\int_{(a)}(\theta-\theta_m)x\,dx-\int_{(a)}(\theta-\theta_m)\,dx\right\}=0 \quad \ldots\ldots..(5)$$

Here $I=\int_{(a)}x^2\,dx$ is the moment of inertia of the section about Neutral Axis.

$G=\int_{(a)}x\,dx$ is geometrical moment of the section about Neutral Axis.

As all the integrals in the eq. (5) are referred to the Neutral Axis, the eq. (5) will be an implicit function determining the position of the Neutral Axis. When the temperature gradient curve has a point symmetry with respect to a point on middle axis (as in **Fig. 4**) even if the said curve may not be linear, the middle axis coincides with Neutral Axis. We have G about the middle axis$=0$. And θ_m coincides with the mean temperature of the section so the equation (5) is identically satisfied.

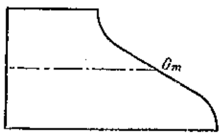


Fig. 4.

In the more complicated distributions of temperature, Neutral Axis is not situated on middle axis and its position shifts by different amount of bending in general.

By this, we see our calculation is fitted, in a strict sense, only for the case of linear distribution of temperature or for the cases of distributions as are shown in **Fig. 4,** at best. We might, however, apply it as an approximate solution of the case concerning temperature distributions not so far deviating from a straight line. In the case of reinforced concrete beams we have to adapt the treatment needed in that line for finding the position of Neutral Axis (see my paper: Über die Wärmeleitung und die Berechnung von Wärmespannungen in Eisenbetonstützmauern u. s. w. Beton und Eisen Heft 21 s. 385, 1926).

Giving one more example, a beam is supported so as to elongate freely and is provided with end moments M_0 as in **Fig. 5.**

Its bottom face is heated uniformly.

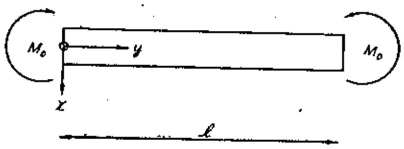


Fig. 5.

$$-\frac{d^2\xi}{dy^2}=\frac{M_0}{EI}+\frac{\Phi}{I}$$

By this equation we see directly that the curvature is constant at any point of the beam, as the right hand side member is constant.

Integrating this equation with boundary conditions $\xi=0$ at both ends, we obtain:—

$$-\xi=\left(\frac{M_0}{EI}+\frac{\Phi}{I}\right)\left(\frac{y^2}{2}-\frac{l}{2}y\right)$$

Hence we can make $\xi=0$ at any point of the beam, putting $M_0=-E\Phi$.

If we put $M_0=0$,

$$-\xi=\frac{\Phi}{I}\left(\frac{y^2}{2}-\frac{l}{2}y\right), \quad \text{parabola or circular arc} \ldots\ldots(6)$$

and

$$-\frac{d^2\xi}{dy^2}=\frac{\Phi}{I}$$

Just the same result as in the case of cantilever follows.

II. Equations of Thermal Flexure of Thin Plates.

We assume that as is discussed in the previous case of beams, plates are provided sufficiently with expansion joints and can shift freely on their original plane without hinderances to their boundaries.(1) And also we assume that neither they are too thick nor too thin and that the deflection caused by thermal bending is not extraordinary large. In a word, all the conditions are appropriate with so-called "thin plates" in Mechanics. And it is assumed that Kirchhoff's condition of plate bending holds well:— the normal to the Neural Plane remains normal straight line to the elastic surface after the plate being bended. $y-z$ coordinate plane is taken on the original Neutral Plane of the plate and relative displacements along y and z of the point C with respect to the point B are expressed by η and ζ resp. (**Fig. 6**), the normal strains parallel with y and z are respectively as follows:—

(1) If supports are rigid, the direct stresses must be taken into considerations.

$$\frac{\partial\eta}{\partial y}=\epsilon_y+\alpha(\theta-\theta_m),\quad \frac{\partial\zeta}{\partial z}=\epsilon_z+\alpha(\theta-\theta_m)\ \cdots\cdots\cdots\cdots(1)^{(1)}$$

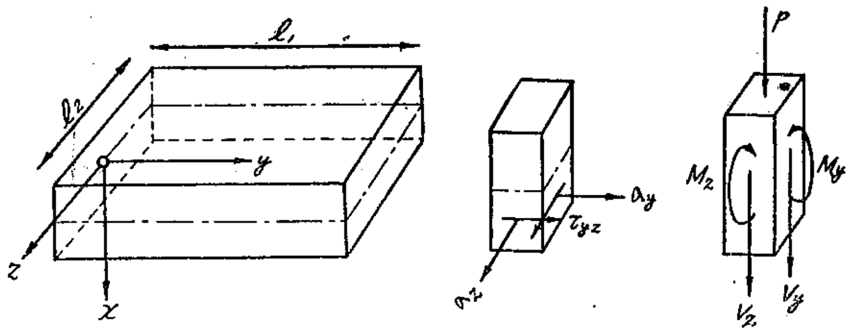


Fig. 6.

And the shearing strain is not affected by the temperature:—

$$\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}=\gamma_{yz}\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots(2)^{(1)}$$

According to Hooke's Law of stress and strain,

$$\left.\begin{aligned}\epsilon_y&=\frac{1}{E}\left(\sigma_y-\frac{1}{m}\sigma_z\right)\\ \epsilon_z&=\frac{1}{E}\left(\sigma_z-\frac{1}{m}\sigma_y\right)\\ \gamma_{yz}&=\frac{1}{G}\tau_{yz}\end{aligned}\right\}$$

here σ_y, σ_z are normal stresses along y and z resp. and τ_{yz} is a shearing stress along y on the plane perpendicular to z (**Fig. 6.**), m is the Poisson's number, E and G are Young's Modulus and Rigidity Modulus resp. These are all assumed to be constants independent of temperature, and

$$G=\frac{E}{2\left(1+\frac{1}{m}\right)}$$

From geometrical relations (**Fig. 1.**)

$$\eta=-x\frac{\partial\xi}{\partial y},\qquad \zeta=-x\frac{\partial\xi}{\partial z}$$

or
$$\frac{\partial\eta}{\partial y}=-x\frac{\partial^2\xi}{\partial y^2},\quad \frac{\partial\zeta}{\partial z}=-x\frac{\partial^2\xi}{\partial z^2},\quad \gamma_{yz}=-2x\frac{\partial^2\xi}{\partial y\,\partial z}$$

Putting these relations in the above equations (1) & (2), and solving them

(1) H. Lorenz: tech. Elastiz. s. 583, or Föppl: tech. Mech. V. s. 236.

with respect to σ_y, σ_z and τ_{yz},

$$\left.\begin{aligned}
\sigma_y&=-\frac{E}{1-\dfrac{1}{m^2}}\left\{x\frac{\partial^2\xi}{\partial y^2}+\frac{x}{m}\frac{\partial^2\xi}{\partial z^2}+\alpha\left(1+\frac{1}{m}\right)(\theta-\theta_m)\right\}\\
\sigma_z&=-\frac{E}{1-\dfrac{1}{m^2}}\left\{x\frac{\partial^2\xi}{\partial z^2}+\frac{x}{m}\frac{\partial^2\xi}{\partial y^2}+\alpha\left(1+\frac{1}{m}\right)(\theta-\theta_m)\right\}\\
\tau_{yz}&=-2xG\frac{\partial^2\xi}{\partial y\,\partial z}
\end{aligned}\right\}\quad\cdots(3)$$

If we take moments of these stresses about Neutral surface,

$$\left.\begin{aligned}
M_y&=\int_{(a)}\sigma_y x\,dx=-\frac{EI}{1-\dfrac{1}{m^2}}\left\{\frac{\partial^2\xi}{\partial y^2}+\frac{1}{m}\frac{\partial^2\xi}{\partial z^2}\right\}-\frac{E\alpha}{1-\dfrac{1}{m}}\int_{(a)}(\theta-\theta_m)\,x\,dx\\
M_z&=\int_{(a)}\sigma_z x\,dx=-\frac{EI}{1-\dfrac{1}{m^2}}\left\{\frac{\partial^2\xi}{\partial z^2}+\frac{1}{m}\frac{\partial^2\xi}{\partial y^2}\right\}-\frac{E\alpha}{1-\dfrac{1}{m}}\int_{(a)}(\theta-\theta_m)\,x\,dx\\
M_{yz}&=\int_{(a)}\tau_{yz}x\,dx=-\left(1-\frac{1}{m}\right)\frac{EI}{1-\dfrac{1}{m^2}}\frac{\partial^2\xi}{\partial y\,\partial z}
\end{aligned}\right\}(4)$$

$\dfrac{EI}{1-\dfrac{1}{m^2}}$ is the flexural rigidity of a plate and we express it symbolically by D and $\alpha\int_{(a)}(\theta-\theta_m)\,x\,dx$ by Φ as in the case of a beam,

$$\left.\begin{aligned}
M_y&=-D\left\{\frac{\partial^2\xi}{\partial y^2}+\frac{1}{m}\frac{\partial^2\xi}{\partial z^2}\right\}-\frac{E\Phi}{1-\dfrac{1}{m}}\\
M_z&=-D\left\{\frac{\partial^2\xi}{\partial z^2}+\frac{1}{m}\frac{\partial^2\xi}{\partial y^2}\right\}-\frac{E\Phi}{1-\dfrac{1}{m}}\\
M_{yz}&=-\left(1-\frac{1}{m}\right)D\frac{\partial^2\xi}{\partial y\,\partial z}
\end{aligned}\right\}\cdots\cdots\cdots(4)_a$$

As Φ depends only upon the value of x, the shears have no terms about Φ:

$$\left.\begin{aligned}
V_y&=\frac{\partial M_y}{\partial y}+\frac{\partial M_{yz}}{\partial z}\\
V_z&=\frac{\partial M_z}{\partial z}+\frac{\partial M_{yz}}{\partial y}
\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots(5)$$

Reactions on the boundaries:—

$$\left.\begin{aligned} R_y&=\left[V_y+\frac{\partial M_{yz}}{\partial z}\right]_{y=0 \text{ or } l_1} \\ R_z&=\left[V_z+\frac{\partial M_{yz}}{\partial y}\right]_{z=\pm\frac{l_2}{2}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (6)$$

And the equation of equilibrium is,

$$\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}+p=0$$

Or symbolically

$$\Delta\Delta\xi=\frac{p}{D}$$

where p is a distributed load on plate.

Δ means Laplacian operator i.e. $\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

or $$\Delta\Delta\equiv\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)^2$$

Taking the thermal bending only, $p=0$

$$\Delta\Delta\xi=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)^{(1)}$$

This homogeneous equation is the equation of the elastic surface of a plate without distributed load. We see, therefore, immediately that every point of the plate remains in its original plane under one face heating on condition that all its boundaries are kept in its original plane, (we call this, provisionally, *encastré* condition though it is a little different from the case usually called) this fact will be later put to experimental test.

In the case that all the boundaries are *encastré* we have

$$\left.\begin{aligned} &\sigma_y=\sigma_z=-\frac{E\alpha}{1-\frac{1}{m}}(\theta-\theta_m), \qquad \tau_{yz}=0 \\ &M_y=M_z=-\frac{E\Phi}{1-\frac{1}{m}}, \qquad M_{yz}=0 \\ &V_y=V_z=0 \\ &R_y=R_z=0 \end{aligned}\right\} \quad \ldots\ldots\ldots (8)$$

In my paper: Über die Wärmeleitung, etc. (loc. cit.) I have assumed that

(1) A more general equation, though the present author was not ware of it till he completed this computation, was deduced by Dr. Ing. Nádai, Göttingen, which is applied for the cases of non-uniform distribution of temperature (Nádai: Elastische Platten 1925, S. 268).

the wall surface remains a plane and have treated the wall as a beam. Here we have found a justification of it, at least, in the above mentioned case.

As to the assumption as a beam, we have usually high Poisson's number for the porous materials like concrete and we might neglect the term $\frac{1}{m}$ (a lateral effect) at the sacrifice of rigorousness, which commits no greater errors than those to be made in the usual calculations of stresses in concrete or reinforced concrete structures.

III. Infinitely extended Strips or Bands.

If we consider temperature effect only, $p=0$, and the foundamental equation (7) degenerates into the ordinary equation:

$$\frac{d^4\xi}{dy^4}=0 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

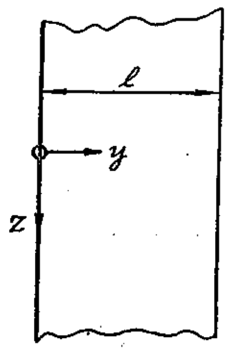


Fig. 7.

Its general solution

$$\xi=C_1 y^3+C_2 y^2+C_3 y+C_4$$

If both ends are supported so as to hold the tangents to the Neutral surface in their original plane (like the condition of *encastré* in the usual bending of plates), for the boundary conditions to determine C_1, C_2, C_3 and C_4 we have

$$\xi=0 \quad \text{and} \quad \frac{d\xi}{dy}=0 \quad \text{at} \quad y=0 \text{ and } l$$

This gives $C_1=C_2=C_3=C_4=0$, i. e., ξ is throughout zero or the strip will not bend at all.

$$\xi=0^{(1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Therefore we have,

$$\left.\begin{array}{ll} \sigma_y=\sigma_z=-\dfrac{E}{1-\dfrac{1}{m}}\,\alpha\,(\theta-\theta_m), & \tau_{yz}=0 \\ M_y=M_z=-\dfrac{E\Phi}{1-\dfrac{1}{m}}, & M_{yz}=0 \\ V_y=V_z=0 & \end{array}\right\} \ldots\ldots (3)$$

If both ends are free to rotate (like the *simply supported* condition in the usual bending of plates), for the boundary conditions we have:—

1) This is not true, of course, if both ends are supported not to yield to in y direction.

$$\xi=0 \quad \text{and} \quad M_y=-D\frac{d^2\xi}{dy^2}-\frac{E\Phi}{1-\frac{1}{m}}=0 \quad \text{at} \quad y=0 \text{ and } l$$

These give:

$$C_1=0, \quad C_2=-\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}, \quad C_3=\frac{E\Phi l}{2D\left(1-\frac{1}{m}\right)} \quad \text{and} \quad C_4=0$$

$$\xi=-\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}(y^2-ly) \ldots \text{parabola or circular arc, if } \xi \text{ is small.} \qquad \ldots\ldots(4)$$

$$M_y=0, \quad M_z=-E\Phi, \quad M_{yz}=0, \quad V_y=V_z=0 \ldots\ldots(5)$$

If one end is encastré and the other end left entirely free, for boundary conditions we have:—

$$\xi=0 \quad \text{and} \quad \frac{d\xi}{dy}=0 \quad \text{at} \quad y=0$$

$$M_y=-D\frac{d^2\xi}{dy^2}-\frac{E\Phi}{1-\frac{1}{m}}=0 \quad \text{and} \quad R_y=-D\frac{d^3\xi}{dy^3}=0 \quad \text{at} \quad y=l$$

These give: $C_1=0, \quad C_2=-\dfrac{E\Phi}{2D\left(1-\frac{1}{m}\right)}, \quad C_3=0, \quad C_4=0$

$$\xi=-\frac{E\Phi y^2}{2D\left(1-\frac{1}{m}\right)} \quad \text{parabola or circular arc, if } \xi \text{ is small} \ldots(6)$$

$$M_y=0, \quad M_z=-E\Phi, \quad M_{yz}=0 \ldots\ldots\ldots(7)$$

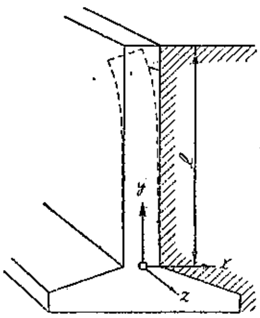


Fig. 8.

This is applicable for the retaining wall without buttress as is shown in **Fig. 8.** The bending moment exists only lengthwise z-direction, along which the wall is not allowed to deform freely. As to stresses, σ_y vanishes or nearly vanishes, (for linear distribution of temperature or for other distributions) and σ_z has the following value:—

$$\sigma_z=\frac{E^2\Phi x}{m\left(1-\frac{1}{m}\right)\left(1-\frac{1}{m^2}\right)D}-\frac{E\alpha(\theta-\theta_m)}{1-\frac{1}{m}}$$

$$=\frac{E\alpha\int_{(a)}(\theta-\theta_m)\,xdx}{m\left(1-\frac{1}{m}\right)\int_{(a)}x^2\,dx}x-\frac{E\alpha\,(\theta-\theta_m)}{1-\frac{1}{m}}\quad\ldots\ldots\ldots\ldots\ldots(8)$$

In the case of uniformly varying distribution of temperature, (**Fig. 2.**)

$$\theta-\theta_m=\Theta x,\qquad \Theta=\frac{\theta_2-\theta_1}{a},\qquad \text{constant.}$$

$$\sigma_z=-E\alpha\Theta x=-E\alpha\,(\theta-\theta_m)\ldots\ldots\ldots\ldots\ldots\ldots(8)_a$$

This coincides exactly with the case of a straight beam, therefore we may calculate the normal stresses lengthwise the wall simply as a straight beam. Here we have found one more justified case of my former calculations about the temperature stresses in a reinforced concrete wall.

In a plain concrete wall we have approximately $E=200\,000$ kg/cm², $\alpha=.00001$ for 1°C, for 100°C difference of temperature between outside and middle plane, we obtain $\sigma_z=\pm 200$ kg/cm² on the outside surface. It is a dangerous value, not to speak of the tension, even for the compression of concrete. In the case of a reinforced concrete wall, I computed them in my former paper. (Beton und Eisen: Heft 21. s. 385, 1926, loc. cit.)

IV. Rectangular Plates with Four Sides Free to Rotate.

The general solution of a bi-harmonic equation (**II.** 7) is well known as the following form:—

$$\xi=f_1\,(y+iz)+f_2\,(y-iz)+(y^2+z^2)\{f_3\,(y+iz)+f_4\,(y-iz)\}$$

Here f_1, f_2, f_3 and f_4 are any analytical functions. But it may not be easy to find these proper functions for the present conditions. The usual Navier's Method in using ΣΣ sin. sin. is not convenient here neither. It seems to me that the method of M. Lévy may here be employed with efficacy, though our present equation is a homogeneous equation in place of a non-homogeneous one which is usually treated by this method.

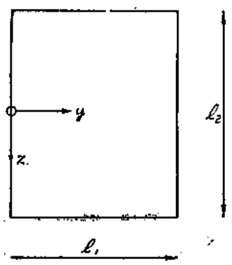


Fig. 9.

Put $$\xi=\xi'+\xi''$$

We may use as ξ' a solution of an infinitly extended strip obtained in the preceeding paragraph, i. e., $\xi'=\dfrac{E\Phi}{2D\left(1-\frac{1}{m}\right)}(l_1y-y^2)$

As to ξ'' we take the following form

$$\xi'' = \sum_s Z_s \sin \frac{s\pi y}{l_1}$$

here Z_s is a function of z only and s is any positive integer.

Therefore,

$$\xi = \xi' + \xi'' = \frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}(l_1 y - y^2) + \sum_s Z_s \sin \frac{s\pi y}{l_1} \quad \ldots\ldots(1)$$

This function ξ must satisfy the fundamental eq.

$$\Delta\Delta\xi = 0 \quad \text{or} \quad \Delta\Delta\xi' + \Delta\Delta\xi'' = 0$$

and the following boundary conditions:

$$\left.\begin{aligned} y=0 \ \&\ l_1: &\quad \xi = 0 \quad \text{and} \quad M_y = 0 \\ z = \pm\frac{l_2}{2}: &\quad \xi = 0 \quad \text{and} \quad M_z = 0 \end{aligned}\right\}$$

As $\Delta\Delta\xi' = 0$, we have $\Delta\Delta\xi'' = 0$ i. e.

$$\frac{d^4 Z_s}{dz^4} - \frac{2s^2\pi^2}{l_1^2}\frac{d^2 Z_s}{dz^2} + \frac{s^4\pi^4}{l_1^4} Z_s = 0$$

The general solution of this equation is,

$$Z_s = (C_s + C_s' z)\, e^{\frac{s\pi z}{l_1}} + (D_s + D_s' z)\, e^{-\frac{s\pi z}{l_1}}$$

$$= A_s \cosh \frac{s\pi z}{l_1} + B_s \frac{s\pi z}{l_1} \sinh \frac{s\pi z}{l_1} + C_s \sinh \frac{s\pi z}{l_1} + D_s \frac{s\pi z}{l_1} \cosh \frac{s\pi z}{l_1}$$

here A_s, B_s, C_s and D_s are arbitrary constants.

In the present boundary conditions we take only the symmetrical terms,

$$Z_s = A_s \cosh \frac{s\pi z}{l_1} + B_s \frac{s\pi z}{l_1} \sinh \frac{s\pi z}{l_1}$$

Therefore, $$\xi'' = \sum_s \left(A_s \cosh \frac{s\pi z}{l_1} + B_s \frac{s\pi z}{l_1} \sinh \frac{s\pi z}{l_1} \right) \sin \frac{s\pi y}{l_1} \quad \ldots\ldots(2)$$

At $y=0$ & l_1: $\xi' = 0$ and $\xi'' = 0$ $\therefore$ $\xi = 0$ is satisfied

$$M_y = -D\left\{\frac{\partial^2\xi'}{\partial y^2} + \frac{1}{m}\frac{\partial^2\xi'}{\partial z^2}\right\} - D\left\{\frac{\partial^2\xi''}{\partial y^2} + \frac{1}{m}\frac{\partial^2\xi''}{\partial z^2}\right\} - \frac{E\Phi}{1-\frac{1}{m}}$$

$$= \frac{E\Phi}{1-\frac{1}{m}} - D\left\{-\sum \frac{s^2\pi^2}{l_1^2} Z_s \sin \frac{s\pi y}{l_1} + \frac{1}{m}\sum \frac{d^2 Z_s}{dz^2} \sin \frac{s\pi y}{l_1}\right\} - \frac{E\Phi}{1-\frac{1}{m}}$$

$$= -D\sum\left\{-\frac{s^2\pi^2}{l_1^2} Z_s + \frac{1}{m}\frac{d^2 Z_s}{dz^2}\right\}\sin \frac{s\pi y}{l_1}$$

$$\therefore \qquad M_y\Big]_{y=0\ \&\ l_1} = 0$$

At. $Z=\pm\frac{l_2}{2}$ we must determine A_s & B_s so as to give $\xi=0$ and $M_z=0$

$$\xi\Big]_{z=\pm\frac{l_2}{2}}=(\xi'+\xi'')\Big]_{z=\pm\frac{l_2}{2}}=0$$

or
$$\sum_s\left(A_s\cosh\frac{s\pi l_2}{2l_1}+B_s\frac{s\pi l_2}{2l_1}\sinh\frac{s\pi l_2}{2l_1}\right)\sin\frac{s\pi y}{l_1}=-\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}(l_1y-y^2)$$

If we express the right hand member in Fourier's Sine-Series

$$\sum_s\frac{8l_1^2}{s^3\pi^3}C\sin\frac{s\pi y}{l_1}=C\,(l_1y-y^2)\qquad s=1,\ 3,\ 5\ldots.$$

therefore we have one equation to determine A_s and B_s as follows:

$$A_s\cosh\frac{s\pi l_2}{2l_1}+B_s\frac{s\pi l_2}{2l_1}\sinh\frac{s\pi l_2}{2l_1}=\frac{8l_1^2}{s^3\pi^3}C\qquad\ldots\ldots\ldots.(3)$$

$$s=1,\ 3,\ 5\ldots.$$

$$C=-\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}$$

Next condition is $M_z\Big]_{z=\frac{l_2}{2}}=0$

$$M_z=-D\left\{\frac{\partial^2\xi'}{\partial z^2}+\frac{1}{m}\frac{\partial^2\xi'}{\partial y^2}\right\}-D\left\{\frac{\partial^2\xi''}{\partial z^2}+\frac{1}{m}\frac{\partial^2\xi''}{\partial y^2}\right\}-\frac{E\Phi}{1-\frac{1}{m}}$$

$$=\frac{E\Phi}{m\left(1-\frac{1}{m}\right)}-D\sum\left\{\frac{d^2Z_s}{dz^2}-\frac{Z_s}{m}\frac{s^2\pi^2}{l_1^2}\right\}\sin\frac{s\pi y}{l_1}-\frac{E\Phi}{1-\frac{1}{m}}$$

$$=-D\sum\left\{\frac{d^2Z_s}{dz^2}-\frac{Z_s}{m}\frac{s^2\pi^2}{l_1^2}\right\}\sin\frac{s\pi y}{l_1}-\left(1-\frac{1}{m}\right)\frac{E\Phi}{1-\frac{1}{m}}$$

$M_z\Big]_{z=\frac{l_2}{2}}=0$ gives;

$$\sum\left\{A_s\frac{s^2\pi^2}{l_1^2}\cosh\frac{s\pi l_2}{2l_1}+B_s\frac{s^2\pi^2}{l_1^2}\left(2\cosh\frac{s\pi l_2}{2l_1}+\frac{s\pi l_2}{2l_1}\sinh\frac{s\pi l_2}{2l_1}\right)\right.$$
$$\left.-\frac{1}{m}\frac{s^2\pi^2}{l_1^2}\left(A_s\cosh\frac{s\pi l_2}{2l_1}+B_s\frac{s\pi l_2}{2l_1}\sin\frac{s\pi l_2}{2l_1}\right)\right\}\sin\frac{s\pi y}{l_1}=2\left(1-\frac{1}{m}\right)C$$

Expressing right hand side member in Fourier's Sine-series

$$\sum\frac{8\left(1-\frac{1}{m}\right)C}{\pi s}\sin\frac{s\pi y}{l_1}=2\left(1-\frac{1}{m}\right)C\qquad s=1,\ 3,\ 5\ldots.$$

Therefore we have another equation to determine A_s & B_s.

$$A_s \cosh \frac{s\pi l_2}{2l_1} + B_s\left\{\frac{2}{1-\frac{1}{m}} \cosh \frac{s\pi l_2}{2l_1} + \frac{s\pi l_2}{2l_1} \sinh \frac{s\pi l_2}{2l_1}\right\} = \frac{8l_1^2}{s^3\pi^3} C \quad ..(4)$$

$$s=1,\ 3,\ldots \qquad C = -\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}$$

From equation (3) & (4) we have:

$$A_s = \frac{8l_1^2}{s^3\pi^3} \frac{C}{\cosh \frac{s\pi l_2}{2l_1}} \qquad \text{and} \qquad B_s = 0$$

$$\therefore \qquad \xi'' = -\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)} \frac{8l_1^2}{\pi^3} \sum_s \frac{1}{s^3} \frac{\cosh \frac{s\pi z}{l_1}}{\cosh \frac{s\pi l^2}{2l_1}} \sin \frac{s\pi y}{l_1} \quad \ldots.(5)$$

Therefore,

$$\xi = \frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\left\{(l_1 y - y^2) - \frac{8l_1^2}{\pi^3} \sum_{s=1,3\ldots} \frac{1}{s^3} \frac{\cosh \frac{s\pi z}{l_1}}{\cosh \frac{s\pi l_2}{2l_1}} \sin \frac{s\pi y}{l_1}\right\} \quad ..(6)$$

or by
$$\frac{8l_1^2}{\pi^3} \sum_{s=1,3\ldots} \frac{1}{s^3} \sin \frac{s\pi y}{l_1} = l_1 y - y^2$$

We may write it down in uniform style:

$$\xi = \frac{E\Phi}{2D\left(1-\frac{1}{m}\right)} \frac{8l_1^2}{\pi^3} \sum_{s=1,3\ldots} \frac{1}{s^3} \left(1 - \frac{\cosh \frac{s\pi z}{l_1}}{\cosh \frac{s\pi l_2}{2l_1}}\right) \sin \frac{s\pi y}{l_1} \quad \ldots.(7)$$

This solution is not convenient to obtain reactions on sides, as it is comparatively low order with respect to s and the Fourier's series may or may not converge for the higher derivatives. Putting off all these difficulties for the further discussions, we, for the present, proceed to obtain moments, shears and reactions in the formal way.

$$M_y = -D\left\{\frac{\partial^2 \xi}{\partial y^2} + \frac{1}{m}\frac{\partial^2 \xi}{\partial z^2}\right\} - \frac{E\Phi}{1-\frac{1}{m}}$$

$$= -E\Phi \frac{4}{\pi} \sum_{s=1,3\ldots} \frac{1}{s} \frac{\cosh \frac{s\pi z}{l_1}}{\cosh \frac{s\pi l_2}{2l_1}} \sin \frac{s\pi y}{l_1}$$

$$\left.\begin{aligned}
M_z &= -D\left\{\frac{\partial^2\xi}{\partial z^2}+\frac{1}{m}\frac{\partial^2\xi}{\partial y^2}\right\}-\frac{E\Phi}{1-\dfrac{1}{m}} \\
&= E\Phi\left\{-1+\frac{4}{\pi}\sum\frac{1}{s}\frac{\cosh\dfrac{s\pi z}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}}\sin\frac{s\pi y}{l_1}\right\} \\
M_{yz} &= E\Phi\frac{4}{\pi}\sum\frac{1}{s}\frac{\sinh\dfrac{s\pi z}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}}\cos\frac{s\pi y}{l_1}
\end{aligned}\right\}\quad\cdots\cdots\cdots(8)$$

$$\left.\begin{aligned}
V_y &= -D\left\{\frac{\partial^3\xi}{\partial y^3}+\frac{\partial^3\xi}{\partial y\partial z^2}\right\}=0 \\
V_z &= -D\left\{\frac{\partial^3\xi}{\partial z^3}+\frac{\partial^3\xi}{\partial y^2\partial z}\right\}=0
\end{aligned}\right\}\quad\cdots\cdots\cdots(9)$$

$$\left.\begin{aligned}
R_y &= \left[V_y+\frac{\partial M_{yz}}{\partial z}\right]_{y=l_1}=-E\Phi\frac{4}{l_1}\sum\frac{\cosh\dfrac{s\pi z}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}} \\
R_z &= \left[V_z+\frac{\partial M_{yz}}{\partial y}\right]_{z=\frac{l_2}{2}}=-E\Phi\frac{4}{l_1}\sum\frac{\sinh\dfrac{s\pi z}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}}\sin\frac{s\pi y}{l_1}
\end{aligned}\right\}\quad\cdots(10)$$

Discussions on M, V and R: M_y vanishes on z-sides and has the constant value $-E\,\Phi$ on y-sides, which is the largest bending moment. M_z vanishes on y-sides and has the constant value $-E\,\Phi$ on z-sides. The surfaces represent M_y and M_z are of an anticlastic form. (**Fig. 10.**)

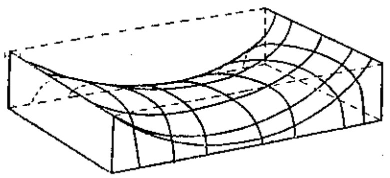


Fig. 10.

Take the sum of M_y and M_z,

$$M_y+M_z=-E\Phi \quad , \quad \text{constant.}$$

Therefore

$$\frac{\partial^2\xi}{\partial y^2}+\frac{\partial^2\xi}{\partial z^2}=-\frac{M_y+M_z}{D\left(1+\dfrac{1}{m}\right)}$$

$$-\frac{2E\Phi}{D\left(1+\dfrac{1}{m}\right)\left(1-\dfrac{1}{m}\right)}=-\frac{E\Phi}{D\left(1-\dfrac{1}{m}\right)} \quad , \quad \text{constant.}$$

The sum of bending moments together with Eulerian Mean Curvature is constant at every point of the plate.

M_{yz} vanishes on the central lines of the plate. On y-side $(y=l_1)$

$$M_{yz}=-E\Phi\frac{4}{\pi}\sum_s\frac{1}{s}\frac{\sinh\frac{s\pi z}{l_1}}{\cosh\frac{s\pi l_2}{2l_1}}$$

It is convergent except at the corner where $\frac{\sinh}{\cosh}$ becomes equal to $\tanh\frac{s\pi l_2}{2l_1}$, which is nearly equal to 1 for higher value of s. And the series approaches to the harmonic series which diverges. On z-side $\left(z=\frac{l_2}{2}\right)$

$$M_{yz}=E\Phi\frac{4}{\pi}\sum_s\frac{1}{s}\tanh\frac{s\pi l_2}{2l_1}\cos\frac{s\pi y}{l_1}$$

Though this series has the vibratory property, still it converges except at the corner where it diverges as is stated above.

Regarding V_y and V_z, we see that they vanish at every point of the plate from the geometrical relation that the Eulerian Mean Curvature is constant everywhere i. e.:—

$$\Delta\xi=-\frac{E\Phi}{D\left(1-\frac{1}{m}\right)}, \qquad \text{constant}$$

$$V_y=-D\frac{\partial}{\partial y}\Delta\xi=0$$

$$V_z=-D\frac{\partial}{\partial z}\Delta\xi=0$$

V_y & V_z are zero at every point of the plate, so we practically need not obtain R_y and R_z, in case of M_{yz} on sides are known. All the required values of external agents on the boundaries are, therefore, to be said to be settled down, in case of the values of M_{yz} on sides are known. About the convergency & divergency of R_y, following the usual criterion,

we put $$u_s=\frac{\cosh\frac{s\pi z}{l_1}}{\cosh\frac{s\pi l_2}{2l_1}}$$

$$u_{s+1}=\frac{\cosh\frac{(s+2)\pi z}{l_1}}{\cosh\frac{(s+2)l_2}{2l_1}}=\frac{\cosh\frac{s\pi z}{l_1}\cosh\frac{2\pi z}{l_1}+\sinh\frac{s\pi z}{l_1}\sinh\frac{2\pi z}{l_1}}{\cosh\frac{s\pi l_2}{2l_1}\cosh\frac{\pi l_2}{l_1}+\sinh\frac{s\pi l_2}{2l_1}\sinh\frac{\pi l_2}{l_1}}$$

$$\frac{u_{s+1}}{u_s}=\frac{\cosh\frac{2\pi z}{l_1}+\tanh\frac{s\pi z}{l_1}\sinh\frac{2\pi z}{l_1}}{\cosh\frac{\pi l_2}{l_1}+\tanh\frac{s\pi l_2}{2l_1}\sinh\frac{\pi l_2}{l_1}}$$

When the value of z is less than $\frac{l_2}{2}$ this ratio is less than a constant value which is less than 1. Therefore the series converges for $z<\frac{l_2}{2}$ i. e., R_y has a finite value except near the corners of a plate.

As to R_z, it is vibratory and in consequence is difficult to get to a reasonable value. If we take, however, a parallel line a little inside from the side $z=\frac{l_2}{2}$, it is easy to prove that the series will converge on this line except near the corners of a plate. Hardly we need treat R_z as such. If we interchange the y-side with z, we can obtain R_z in the form of R_y.

Numerical Example: (a square plate)

We take, for a numerical example, a case of square plate.

Put $l_1=l_2=l$

$$\left.\begin{aligned}\xi&=\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\frac{8l^3}{\pi^3}\sum_{s=1,3,\ldots}\frac{1}{s^3}\left\{1-\frac{\cosh\frac{s\pi z}{l}}{\cosh\frac{s\pi}{2}}\right\}\sin\frac{s\pi y}{l}\\ \text{or}\quad \xi&=K_1\frac{8l^3}{\pi^3}\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\end{aligned}\right\}\quad\ldots(11)$$

This series is quickly convergent and it may be sufficient if we take first 2 terms only.

The value of K_1 in eq. (11) (see **Pl. I.**)

$z \backslash y$	0	$\frac{l}{12}$	$2\frac{l}{12}$	$3\frac{l}{12}$	$4\frac{l}{12}$	$5\frac{l}{12}$	$6\frac{l}{12}$
0	0	.1816	.3374	.4513	.5207	.5558	.5656
$\frac{l}{12}$	0	.1779	.3302	.4413	.5086	.5424	.5518
$2\frac{l}{12}$	0	.1664	.3084	.4110	.4722	.5024	.5106
$3\frac{l}{12}$	0	.1458	.2695	.3573	.4082	.4323	.4385
$4\frac{l}{12}$	0	.1147	.2108	.2773	.3139	.3299	.3337
$5\frac{l}{12}$	0	.0683	.1243	.1619	.1807	.1876	.1889
$6\frac{l}{12}$	0	0	0	0	0	0	0

$$\left.\begin{aligned} M_y &= -E\Phi\frac{4}{\pi}\sum\frac{1}{s}\frac{\cosh\frac{s\pi z}{l}}{\cosh\frac{s\pi}{2}}\sin\frac{s\pi y}{l} \\ \text{or}\qquad M_y &= -K_2 E\Phi \end{aligned}\right\}\quad\ldots\ldots(12)$$

This is quickly convergent for the values of z near 0 i. e., the central band of a plate. The nearer it approaches the corners of a plate (z becomes equal to $\frac{l}{2}$ together with y equal to 0), the more slowly it converges. For example, it may be suffiicient it we take only the first few terms for M_y at $z=0$. The following values in the table are obtained by taking only the first 2 terms for M_y at $z=0$, but we must take 5 terms for values M_y at $y=\frac{l}{12}$ and $z=\frac{l}{2}$ to obtain the same order of accuracy.

$$z=0:\qquad M_y=-E\Phi\frac{4}{\pi}\sum\frac{1}{s}\frac{1}{\cosh\frac{s\pi}{2}}\sin\frac{s\pi y}{l}$$

$$z=\frac{5}{12}l:\quad M_y=-E\Phi\frac{4}{\pi}\sum\frac{1}{s}\frac{\cosh\frac{5s\pi}{12}}{\cosh\frac{s\pi}{2}}\sin\frac{s\pi y}{l}$$

$$z=\frac{l}{2}:\qquad M_y=-E\Phi\frac{4}{\pi}\sum\frac{1}{s}\sin\frac{s\pi y}{l}=-E\Phi$$

The values of K_2 in eq. (12)

y \ z	0	$5\frac{l}{12}$	$6\frac{l}{12}$
$\frac{l}{12}$		.501	1.0
$2\frac{l}{12}$	.26	.704	1.0
$3\frac{l}{12}$		.780	1.0
$4\frac{l}{12}$	.438	.812	1.0
$6\frac{l}{12}$	.497 *	.854	1.0

* This must be rigorously treated equal to .500, as at the center of the plate

$$\left(z=0\ \&\ y=\frac{l}{2}\right);\quad M_y=-E\Phi\frac{4}{\pi}\sum\frac{(-1)^{\frac{s-1}{2}}}{s}\frac{1}{\cosh\frac{s\pi}{2}}=-\frac{1}{2}E\Phi.$$

By $M_z=-E\Phi-M_y$, we obtain the values of M_y from those of M_z.

$$\left.\begin{aligned} M_{yz}&=E\Phi\frac{4}{\pi}\sum\frac{1}{s}\frac{\sinh\frac{s\pi z}{l}}{\cosh\frac{s\pi}{2}}\cos\frac{s\pi y}{l} \\ \text{or}\qquad M_{yz}&=-K_3 E\Phi \end{aligned}\right\}\quad\ldots\ldots\ldots(13)$$

On y side; $\quad y=l$: $\quad M_{yz}=-E\Phi\frac{4}{\pi}\sum\frac{1}{s}\frac{\sinh\frac{s\pi z}{l}}{\cosh\frac{s\pi}{2}}$

This is quickly convergent except near the corners of the plate.

On z side, $\quad z=\frac{l}{2}$: $\quad M_{yz}=E\Phi\frac{4}{\pi}\sum\frac{1}{s}\tanh\frac{s\pi}{2}\cos\frac{s\pi y}{l}$

This is vibratory but converges except at the corners of the plate. We obtained the following values in the table, by taking the first 3 or 4 terms in the case of M_{yz} on y-side and the first 12 terms in some cases of M_{yz} on z-side. These two values, of course, must coincide with each other.

The values of K_3 in eq. (13)

$z \backslash y$	l	$y \backslash z$	$\frac{l}{2}$
0	0	$6\frac{l}{12}$	0
$\frac{l}{12}$	.1407	$5\frac{l}{12}$	— .1375
$2\frac{l}{12}$	.2966	$4\frac{l}{12}$	— .2982
$3\frac{l}{12}$	.4860	$3\frac{l}{12}$	— .4841
$4\frac{l}{12}$	.7450	$2\frac{l}{12}$	— .7391
$5\frac{l}{12}$	1.1636	$\frac{l}{12}$	—1.2279

Remarks:

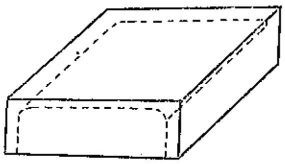


Fig. 11

The results of this paragraph may be applicable for the simply supported flat concrete slab of rectangular form with strong stiffening stems all around 4 sides preventing the sides from being bended but can not resist rotation in the perpendicular direction. (**Fig. 11**). The direct stresses in the slab caused by its un-equal elongation with the stems are neglected. We must expect some difficulties at the corners of the plate.

V. Rectangular Plates with Two Sides Free to Rotate and Two Sides Fixed.

Put, as the foregoing case,

$$\xi=\xi'+\xi''=\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}(l_1y-y^2)$$

$$+\sum_{s=1,3\ldots}\left(A_s\cosh\frac{s\pi z}{l_1}+B_s\frac{s\pi z}{l_1}\sinh\frac{s\pi z}{l_1}\right)\sin\frac{s\pi y}{l_1}\quad\ldots(1)$$

This equation must satisfy the following boundary conditions:

$$\left.\begin{array}{lll} y=0\ \&\ l_1: & \xi=0 \text{ and } & M_y=0 \\ z=\pm\frac{l_2}{2}: & \xi=0 \text{ and } & \frac{\partial\xi}{\partial z}=0 \end{array}\right\}.$$

Already we have seen in the foregoing case that the equation (1) will satisfy the first boundary conditions. Therefore we will determine the constants A_s & B_s from the second boundary conditions, i. e.

$$\xi\Big]_{z=\frac{l_2}{2}}=(\xi'+\xi'')\Big]_{z=\frac{l_2}{2}}=0$$

This gives the same conditions as (3) in the foregoing case, i. e.

$$A_s\cosh\frac{s\pi l_2}{2l_1}+B_s\frac{s\pi l_2}{2l_1}\sinh\frac{s\pi l_2}{2l_1}=\frac{8l_1^2}{s^3\pi^3}C\ \ldots\ldots\ldots(2)$$

where $$C=-\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\quad\text{and}\quad s=1,\ 3\ldots$$

$$\frac{\partial\xi}{\partial z}\Big]_{z=\frac{l_2}{2}}=\left(\frac{\partial\xi'}{\partial z}+\frac{\partial\xi''}{\partial z}\right)\Big]_{z=\frac{l_2}{2}}=0$$

$$\frac{\partial\xi'}{\partial z}\Big]_{z=\frac{l_2}{2}}=0$$

$$\frac{\partial\xi''}{\partial z}\Big]_{z=\frac{l_2}{2}}=\sum\left\{A_s\frac{s\pi}{l_1}\sinh\frac{s\pi l_2}{2l_1}+B_s\left(\frac{s^2\pi^2 l_2}{2l_1}\cosh\frac{s\pi l_2}{2l_1}\right.\right.$$

$$\left.\left.+\frac{s\pi}{l_1}\sinh\frac{s\pi l_2}{2l_1}\right)\right\}\sin\frac{s\pi y}{l_1}=0$$

$$\therefore\quad A_s\sinh\frac{s\pi l_2}{2l_1}+B_s\left(\frac{s\pi l_2}{2l_1}\cosh\frac{s\pi l_2}{2l_1}+\sinh\frac{s\pi l_2}{2l_1}\right)=0\ \ldots(3)$$

From (2) & (3), we obtain

$$\left.\begin{aligned} A_s &= \frac{8l_1^2}{s^3\pi^3} C \frac{1+\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}} \\ B_s &= -\frac{8l_1^2}{s^3\pi^3} C \frac{1}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}} \end{aligned}\right\} \quad \ldots\ldots\ldots(4)$$

$$\therefore\quad \xi'' = -\frac{E\Phi}{2D\left(1-\dfrac{1}{m}\right)}\frac{8l_1^2}{\pi^3}$$

$$\sum\frac{1}{s^3}\frac{\left(1+\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}\right)\cosh\dfrac{s\pi z}{l_1}-\dfrac{s\pi z}{l_1}\sinh\dfrac{s\pi z}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\sin\frac{s\pi y}{l_1} \quad \ldots(5)$$

Therefore

$$\xi = \frac{E\Phi}{2D\left(1-\dfrac{1}{m}\right)}\left\{(l_1y-y^2)-\frac{8l_1^2}{\pi^3}\right.$$

$$\left.\sum_{s=1,3\ldots}\frac{1}{s^3}\frac{\left(1+\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}\right)\cosh\dfrac{s\pi z}{l_1}-\dfrac{s\pi z}{l_1}\sinh\dfrac{s\pi z}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\sin\frac{s\pi y}{l_1}\right\} \quad \ldots(6)$$

or

$$\xi = \frac{E\Phi}{2D\left(1-\dfrac{1}{m}\right)}\frac{8l_1^2}{\pi^3}$$

$$\sum_{s=1,3\ldots}\frac{1}{s^3}\left\{1-\frac{\left(1+\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}\right)\cosh\dfrac{s\pi z}{l_1}-\dfrac{s\pi z}{l_1}\sinh\dfrac{s\pi z}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\right\}\sin\frac{s\pi y}{l_1} \quad \ldots(7)$$

Moments, shears and reactions are as follows:

$$M_y = -E\Phi\frac{4}{\pi}\sum\frac{1}{s}$$

$$\frac{\left(\dfrac{1+\dfrac{1}{m}}{1-\dfrac{1}{m}}+\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}\right)\cosh\dfrac{s\pi z}{l_1}-\dfrac{s\pi z}{l_1}\sinh\dfrac{s\pi z}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\sin\frac{s\pi y}{l_1}$$

$$\left.\begin{aligned}
M_z&=E\Phi\left\{-1+\frac{4}{\pi}\sum\frac{1}{s}\right.\\
&\left.\frac{\left(-\dfrac{1+\dfrac{1}{m}}{1-\dfrac{1}{m}}+\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}\right)\cosh\dfrac{s\pi z}{l_1}-\dfrac{s\pi z}{l_1}\sinh\dfrac{s\pi z}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\sin\frac{s\pi y}{l_1}\right\}\\
M_{yz}&=E\Phi\frac{4}{\pi}\sum\frac{1}{s}\\
&\qquad\frac{\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}\sinh\dfrac{s\pi z}{l_1}-\dfrac{s\pi z}{l_1}\cosh\dfrac{s\pi z}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\cos\frac{s\pi y}{l_1}
\end{aligned}\right\}\quad\ldots\ldots(8)$$

$$\left.\begin{aligned}
V_y&=-\frac{E\Phi}{1-\dfrac{1}{m}}\frac{8}{l_1}\sum\frac{\cosh\dfrac{s\pi z}{l_1}\cos\dfrac{s\pi y}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\\
V_z&=-\frac{E\Phi}{1-\dfrac{1}{m}}\frac{8}{l_1}\sum\frac{\sinh\dfrac{s\pi z}{l_1}\sin\dfrac{s\pi y}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(9)$$

$$\left.\begin{aligned}
R_y&=E\Phi\frac{4}{l_1}\\
&\sum\frac{\left(\dfrac{3-\dfrac{1}{m}}{1-\dfrac{1}{m}}-\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}\right)\cosh\dfrac{s\pi z}{l_1}+\dfrac{s\pi z}{l_1}\sinh\dfrac{s\pi z}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\\
R_z&=-\frac{E\Phi}{1-\dfrac{1}{m}}\frac{8}{l_1}\sum\frac{\sinh\dfrac{s\pi l_2}{2l_1}\sin\dfrac{s\pi y}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}
\end{aligned}\right\}\quad\ldots\ldots(10)$$

If we take the sum of M_y & M_z,

$$M_y+M_z=-E\Phi-\frac{1+\dfrac{1}{m}}{1-\dfrac{1}{m}}E\Phi\frac{8}{\pi}\sum\frac{1}{s}\frac{\cosh\dfrac{s\pi z}{l_1}\sin\dfrac{s\pi y}{l_1}}{\cosh\dfrac{s\pi l_2}{2l_1}+\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}$$

this is not constant as in the foregoing case, but the second term of this right hand side member is quickly convergent, therefore we may conveniently use this relation to obtain the values of M_z from the values of M_y.

As to Eulerian Mean Curvature

$$\Delta\xi = -\frac{E\Phi}{D\left(1-\frac{1}{m}\right)} + \frac{E\Phi}{D\left(1-\frac{1}{m}\right)}\frac{8}{\pi}\sum\frac{1}{s}\frac{\cosh\frac{s\pi z}{l_1}\sin\frac{s\pi y}{l_1}}{\cosh\frac{s\pi l_2}{2l_1}+\frac{s\pi l_2}{2l_1}\operatorname{cosech}\frac{s\pi l_2}{2l_1}}$$

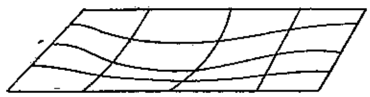


Fig. 12

The first term of this equation is a constant and equal to the foregoing case, i. e., the case of 4 sides free to rotate. The curvature has the opposite sign near the sides $z=\pm\frac{l_2}{2}$ to that of the central portion. (**Fig. 12**)

The value of M_{yz} on sides $z=\pm\frac{l_2}{2}$ vanishes. On $y=l_1$ we have

$$M_{yz} = -E\Phi\frac{4}{\pi}\sum\frac{1}{s}\frac{\frac{s\pi l_2}{2l_1}\coth\frac{s\pi l_2}{2l_1}\sinh\frac{s\pi z}{l_1}-\frac{s\pi z}{l_1}\cosh\frac{s\pi z}{l_1}}{\cosh\frac{s\pi l_2}{2l_1}+\frac{s\pi l_2}{2l_1}\operatorname{cosech}\frac{s\pi l_2}{2l_1}} \quad \ldots.(11)$$

This is zero at $z=0$ & $z=\pm\frac{l_2}{2}$ and has opposite signs on the different sides of $z=0$.

The series for V_y is divergent near the corner and the series for V_z vibrates. If we take a little inside from the sides $z=\pm\frac{l_2}{2}$, it will converge.

Numerical Example: (a square plate)

Put $l_1=l_2=l$

$$\xi = \frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\frac{8l^2}{\pi^3}\sum\frac{1}{s^3}$$

$$\left\{1-\frac{\left(1+\frac{s\pi}{2}\coth\frac{s\pi}{2}\right)\cosh\frac{s\pi z}{l}-\frac{s\pi z}{l}\sinh\frac{s\pi z}{l}}{\cosh\frac{s\pi}{2}+\frac{s\pi}{2}\operatorname{cosech}\frac{s\pi}{2}}\right\}\sin\frac{s\pi y}{l} \quad \ldots.(12)$$

Taking the first 2 terms, we have the values of K_1 in the equation

$$\xi = K_1\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\frac{8l^2}{\pi^3} \quad \ldots\ldots\ldots\ldots\ldots(12)\text{a}$$

as are shown in the following table.

The values of K_1 in eq. $(12)_a$ (see **Pl. II**)

$z \diagdown y$	0	$\frac{l}{12}$	$2\frac{l}{12}$	$3\frac{l}{12}$	$4\frac{l}{12}$	$5\frac{l}{12}$	$6\frac{l}{12}$
0	0	.0621	.108	.1293	.1299	.1216	.117
$\frac{l}{12}$	0	.0594	.103	.1230	.1230	.1146	.110
$2\frac{l}{12}$	0	.0525	.0905	.1067	.1048	.0957	.091
$3\frac{l}{12}$	0	.0402	.0685	.079	.0753	.0663	.062
$4\frac{l}{12}$	0	.0247	.0415	.0466	.0424	.0353	.032
$5\frac{l}{12}$	0	.009	.0145	.0156	.0130	.010	.008
$6\frac{l}{12}$	0	0	0	0	0	0	0

$$\left.\begin{aligned} M_y &= -E\Phi\frac{4}{\pi}\sum\frac{1}{s} \\ &\quad \frac{\left(\dfrac{1+\dfrac{1}{m}}{1-\dfrac{1}{m}}+\dfrac{s\pi}{2}\coth\dfrac{s\pi}{2}\right)\cosh\dfrac{s\pi z}{l}-\dfrac{s\pi z}{l}\sinh\dfrac{s\pi z}{l}}{\cosh\dfrac{s\pi}{2}+\dfrac{s\pi}{2}\operatorname{cosech}\dfrac{s\pi}{2}}\sin\frac{s\pi y}{l} \\ \text{or}\quad M_y &= -K_2\,E\Phi \end{aligned}\right\}\dots\dots(13)$$

As to the Poisson's number m for the materials like concrete, we have not yet a decided value, but we see that it may be a higher value than those of ordinary ductile materials. We may, for the present, assume $m=10$.

The values of K_2 in eq. (13)

$(m=10)$

$y \diagdown z$	0	$5\frac{l}{12}$	$6\frac{l}{12}$
$\frac{l}{12}$	.3343	.9313	1.2055
$2\frac{l}{12}$	.6289	1.078	1.1906
$4\frac{l}{12}$	1.0109	1.143	1.1698
$6\frac{l}{12}$	1.1221	1.149	1.1631

$$\left.\begin{aligned} M_t &= -M_y - E\Phi - \frac{1+\frac{1}{m}}{1-\frac{1}{m}} E\Phi \frac{8}{\pi} \sum \frac{1}{s} \frac{\cosh \frac{s\pi z}{l} \sin \frac{s\pi y}{l}}{\cosh \frac{s\pi}{2} + \frac{s\pi}{2} \operatorname{cosech} \frac{s\pi}{2}} \\ M_z &= -k_3 E\Phi \end{aligned}\right\} \qquad \text{(14)}$$

The values of K_3 in eq. (14)

$y \backslash z$	0	$5\frac{l}{12}$	$6\frac{l}{12}$
$\frac{l}{12}$	.930	1.160	1.824
$2\frac{l}{12}$	.875	1.370	1.670
$4\frac{l}{12}$	.835	1.435	1.550
$6\frac{l}{12}$	.835	1.430	1.424

Regarding M_{yz} on y-side $(y=l)$, we have from the equation (11)

$$\left.\begin{aligned} M_{yz} &= -E\Phi \frac{4}{\pi} \sum \frac{1}{s} \frac{\frac{s\pi}{2} \coth \frac{s\pi}{2} \sinh \frac{s\pi z}{l} - \frac{s\pi z}{l} \cosh \frac{s\pi z}{l}}{\cosh \frac{s\pi}{2} + \frac{s\pi}{2} \operatorname{cosech} \frac{s\pi}{2}} \\ \text{or} \quad M_{yz} &= -K_4 E\Phi \end{aligned}\right\} \qquad \text{(15)}$$

The values of K_4 in eq. (15)

$y \backslash z$	0	$\frac{l}{12}$	$3\frac{l}{12}$	$5\frac{l}{12}$	$6\frac{l}{12}$
l	0	.0984	.2941	.4289	0

$M_{yz}\Big]_{z=\pm\frac{l}{2}} = 0$ for any value of y as mentioned already.

Remarks: The above results may find an important application for a concrete slab bridge with side stems fixed to abutments. (**Fig. 13**).

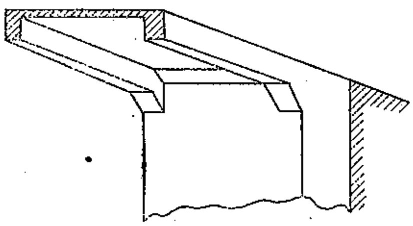


Fig. 13

In the case of a square slab bridge we see from the above results, that the largest bending moments occur near the corners in the longitudinal direction of bridge. They are about twice as large as the bending moments occurring in the flat retaining wall (see the example in **III**)

$$M_z \doteqdot -2E\Phi$$

In the case of uniformly varying distribution of temperature,

$$\theta-\theta_m=\Theta x \quad \text{and} \quad \Phi=\alpha\Theta I. \quad \Theta, \quad \text{const.}$$

$$\int_{(a)} \sigma_z x\, dx = M_z = -2E\alpha\Theta\int_{(a)} x^2\, dx$$

$$\therefore \qquad \sigma_z = -2E\alpha\,(\theta-\theta_m)$$

$$E=200\,000^{\mathrm{kg}}/_{\mathrm{cm}^2}, \quad \alpha=.00001 \text{ for } 1^\circ\text{C. for concrete.}$$

For 50°C difference of temperature between outside & middle planes $\sigma_z=\pm 200^{\mathrm{kg}}/_{\mathrm{cm}^2}$ occurs at the outside surface at the corners of the bridge. It is a dangerous value for ordinary concrete. In the central portion of slab, much less stresses occur as obviously seen in the tables of M_y and M_z, with regard to stress calculations of a reinforced concrete slab, we may put the above values of bending moments equal to the resisting moments of sections as usual practice. Here, of course, we cannot avoid the erroneousness coming from the discrepancy of assumptions for the calculations of bending moments and resisting moments. For the former we assume the body is elastically homogeneous isotropic and for the latter not elastically homogeneous isotropic. This discrepancy is not avoidable even in the usual calculations of bending of reinforced concrete structures under lateral loads, especially in the calculations of slabs and arches. As to the torsion moments, we hardly need consider them, except near the corners as are shown by the equation (15) or the accompanying table. Practically we always put some proper fillets at the corners to increase resistance for them. This is the same for the slabs with different boundary conditions (see the preceeding paragraph etc.).

VI. Rectangular Plates with Two Sides free to Rotate and Two Sides Unsupported.

Put

$$\xi=\xi'+\xi''=\frac{E\Phi}{2D\left(1-\dfrac{1}{m}\right)}(l_1y-y^2)$$

$$+\sum_{s=1,3,\dots}\left(A_s\cosh\frac{s\pi z}{l_1}+B_s\frac{s\pi z}{l_1}\sinh\frac{s\pi z}{l_1}\right)\sin\frac{s\pi y}{l_1} \quad \dots\dots(1)$$

This equation must satisfy the following boundary conditions:

$$\left.\begin{aligned} y=0\ \&\ l_1: &\quad \xi=0 \quad \text{and} \quad M_y=0 \\ z=\pm\frac{l_2}{2}: &\quad M_z=0 \quad \text{and} \quad R_z=0 \end{aligned}\right\}$$

We have seen in the foregoing cases that the equation (1) satisfies the first boundary conditions. Therefore we will determine the constants A_s and B_s by the second conditions, i. e.,

$$M_z\Big]_{z=\frac{l_2}{2}}=0$$

$$A_s\cosh\frac{s\pi l_2}{2l_1}+B_s\left(\frac{2}{1-\frac{1}{m}}\cosh\frac{s\pi l_2}{2l_1}+\frac{s\pi l_2}{2l_1}\sinh\frac{s\pi l_2}{2l_1}\right)=\frac{8l_1^2}{s^3\pi^3}C\ldots.(2)$$

where $C=-\dfrac{E\Phi}{2D\left(1-\frac{1}{m}\right)}$ and $s=1,\ 3,\ 5,\ldots.$

$$R_z\Big]_{z=\frac{l_2}{2}}=-D\left\{\frac{\partial^3\xi}{\partial z^3}+\left(2-\frac{1}{m}\right)\frac{\partial^3\xi}{\partial y^2\,\partial z}\right\}\Big]_{z=\frac{l_2}{2}}=0$$

As $\dfrac{\partial^3\xi'}{\partial z^3}=0$ and $\dfrac{\partial^3\xi'}{\partial y^2\partial z}=0,\quad \therefore\ \dfrac{\partial^3\xi''}{\partial z^3}+\left(2-\dfrac{1}{m}\right)\dfrac{\partial^3\xi''}{\partial y^2\partial z}=0:$

$$\sum_s\left\{-A_s\left(1-\frac{1}{m}\right)\sinh\frac{s\pi l_2}{2l_1}+B_s\left[\left(1+\frac{1}{m}\right)\sinh\frac{s\pi l_2}{2l_1}\right.\right.$$
$$\left.\left.-\left(1-\frac{1}{m}\right)\frac{s\pi l_2}{2l_1}\cosh\frac{s\pi l_2}{2l_1}\right]\right\}\sin\frac{s\pi y}{l_1}=0$$

$$\therefore\quad -A_s\left(1-\frac{1}{m}\right)\sinh\frac{s\pi l_2}{2l_1}+B_s\left\{\left(1+\frac{1}{m}\right)\sinh\frac{s\pi l_2}{2l_1}\right.$$
$$\left.-\left(1-\frac{1}{m}\right)\frac{s\pi l_2}{2l_1}\cosh\frac{s\pi l_2}{2l_1}\right\}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From (2) and (3), we obtain

$$\left.\begin{aligned}A_s&=\frac{8l_1^2}{s^3\pi^3}C\,\frac{\dfrac{1+\frac{1}{m}}{1-\frac{1}{m}}-\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}}{\dfrac{3+\frac{1}{m}}{1-\frac{1}{m}}\cosh\dfrac{s\pi l_2}{2l_1}-\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\\ B_s&=\frac{8l_1^2}{s^3\pi^3}C\,\frac{1}{\dfrac{3+\frac{1}{m}}{1-\frac{1}{m}}\cosh\dfrac{s\pi l_2}{2l_1}-\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\end{aligned}\right\}\ldots\ldots(4)$$

$$\xi=\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\left\{(l_1y-y^2)-\frac{8l_1^2}{\pi^3}\sum_{s=1,3\ldots}\frac{1}{s^3}\right.$$

$$\left.\frac{\left(\frac{1+\frac{1}{m}}{1-\frac{1}{m}}-\frac{s\pi l_2}{2l_1}\coth\frac{s\pi l_2}{2l_1}\right)\cosh\frac{s\pi z}{l_1}+\frac{s\pi z}{l_1}\sinh\frac{s\pi z}{l_1}}{\frac{3+\frac{1}{m}}{1-\frac{1}{m}}\cosh\frac{s\pi l_2}{2l_1}-\frac{s\pi l_2}{2l_1}\operatorname{cosech}\frac{s\pi l_2}{2l_1}}\sin\frac{s\pi z}{l_1}\right\}\quad\ldots\ldots(5)$$

or

$$\xi=\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\frac{8l_1^2}{\pi^3}\sum_{s=1,3\ldots}\frac{1}{s^3}$$

$$\left\{1-\frac{\left(\frac{1+\frac{1}{m}}{1-\frac{1}{m}}-\frac{s\pi l_2}{2l_1}\coth\frac{s\pi l_2}{2l_1}\right)\cosh\frac{s\pi z}{l_1}+\frac{s\pi z}{l_1}\sinh\frac{s\pi z}{l_1}}{\frac{3+\frac{1}{m}}{1-\frac{1}{m}}\cosh\frac{s\pi l_2}{2l_1}-\frac{s\pi l_2}{2l_1}\operatorname{cosech}\frac{s\pi l_2}{2l_1}}\right\}\sin\frac{s\pi y}{l_1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Moments, shears and reactions are as follows:

$$M_y=-E\Phi\frac{4}{\pi}\sum\frac{1}{s}$$

$$\frac{\left(1-\frac{s\pi l_2}{2l_1}\coth\frac{s\pi l_2}{2l_1}\right)\cosh\frac{s\pi z}{l_1}+\frac{s\pi z}{l_1}\sinh\frac{s\pi z}{l_1}}{\frac{3+\frac{1}{m}}{1-\frac{1}{m}}\cosh\frac{s\pi l_2}{2l_1}-\frac{s\pi l_2}{2l_1}\operatorname{cosech}\frac{s\pi l_2}{2l_1}}\sin\frac{s\pi y}{l_1}$$

$$
\left.
\begin{aligned}
M_z &= E\Phi\left\{-1+\frac{4}{\pi}\sum\frac{1}{s}\,
\frac{\left(\dfrac{3+\dfrac{1}{m}}{1-\dfrac{1}{m}}-\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}\right)\cosh\dfrac{s\pi z}{l_1}+\dfrac{s\pi z}{l_1}\sinh\dfrac{s\pi z}{l_1}}{\dfrac{3+\dfrac{1}{m}}{1-\dfrac{1}{m}}\cosh\dfrac{s\pi l_2}{2l_1}-\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\sin\frac{s\pi y}{l_1}\right\}\\
M_{yz} &= E\Phi\,\frac{4}{\pi}\sum\frac{1}{s}\,
\frac{\left(\dfrac{2}{1-\dfrac{1}{m}}-\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}\right)\sinh\dfrac{s\pi z}{l_1}+\dfrac{s\pi z}{l_1}\cosh\dfrac{s\pi z}{l_1}}{\dfrac{3+\dfrac{1}{m}}{1-\dfrac{1}{m}}\cosh\dfrac{s\pi l_2}{2l_1}-\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\cos\frac{s\pi y}{l_1}
\end{aligned}
\right\}\quad\ldots\ldots(7)
$$

$$
\left.
\begin{aligned}
V_y &= \frac{E\Phi}{1-\dfrac{1}{m}}\,\frac{8}{l_1}\sum\frac{\cosh\dfrac{s\pi z}{l_1}\cos\dfrac{s\pi y}{l_1}}{\dfrac{3+\dfrac{1}{m}}{1-\dfrac{1}{m}}\cosh\dfrac{s\pi l_2}{2l_1}-\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\\
V_z &= \frac{E\Phi}{1-\dfrac{1}{m}}\,\frac{8}{l_1}\sum\frac{\sinh\dfrac{s\pi z}{l_1}\sin\dfrac{s\pi y}{l_1}}{\dfrac{3+\dfrac{1}{m}}{1-\dfrac{1}{m}}\cosh\dfrac{s\pi l_2}{2l_1}-\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}
\end{aligned}
\right\}\quad\ldots\ldots(8)
$$

$$
R_y = -E\Phi\,\frac{4}{l_1}
$$

$$\left.\begin{aligned}&\sum\frac{\left(\dfrac{5-\dfrac{1}{m}}{1-\dfrac{1}{m}}-\dfrac{s\pi l_2}{2l_1}\coth\dfrac{s\pi l_2}{2l_1}\right)\cosh\dfrac{s\pi z}{l_1}+\dfrac{s\pi z}{l_1}\sinh\dfrac{s\pi z}{l_1}}{\dfrac{3+\dfrac{1}{m}}{1-\dfrac{1}{m}}\cosh\dfrac{s\pi l_2}{2l_1}-\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}\\ &R_z=0\end{aligned}\right\}\quad\ldots.(9)$$

$$M_y+M_z=-E\Phi+\frac{1+\dfrac{1}{m}}{1-\dfrac{1}{m}}E\Phi\frac{8}{\pi}\sum\frac{1}{s}\frac{\cosh\dfrac{s\pi z}{l_1}\sin\dfrac{s\pi y}{l_1}}{\dfrac{3+\dfrac{1}{m}}{1-\dfrac{1}{m}}\cosh\dfrac{s\pi l_2}{2l_1}-\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}$$

The first term is a constant equal to the case of 4 sides free to rotate. The second term is a quickly convergent series. We may therefore use this equation to obtain M_z from M_y.

As to Eulerian Mean Curvature:—

$$\Delta\xi=-\frac{E\Phi}{D\left(1-\dfrac{1}{m}\right)}-\frac{E\Phi}{D\left(1-\dfrac{1}{m}\right)}\frac{8}{\pi}\sum\frac{1}{s}\frac{\cosh\dfrac{s\pi z}{l_1}\sin\dfrac{s\pi y}{l_1}}{\dfrac{3+\dfrac{1}{m}}{1-\dfrac{1}{m}}\cosh\dfrac{s\pi l_2}{2l_1}-\dfrac{s\pi l_2}{2l_1}\operatorname{cosech}\dfrac{s\pi l_2}{2l_1}}$$

The first term is a constant equal to the case of 4 sides free to rotate. The second term will not change the sign of the curvature. Therefore, we have curvature of the same sense throughout the plate.

The values of M_{yz} and R_y become indefinitely large at the corners.

Numerical Example: (a square plate)

Put $l_1 = l_2 = l$

$$\left.\begin{aligned}
\xi &= \frac{E\Phi}{2D\left(1-\frac{1}{m}\right)} \frac{8l^2}{\pi^3} \sum \frac{1}{s^3} \\
&\left\{1-\frac{\left(\frac{1+\frac{1}{m}}{1-\frac{1}{m}}-\frac{s\pi}{2}\coth\frac{s\pi}{2}\right)\cosh\frac{s\pi z}{l}+\frac{s\pi z}{l}\sinh\frac{s\pi z}{l}}{\frac{3+\frac{1}{m}}{1-\frac{1}{m}}\cosh\frac{s\pi}{2}-\frac{s\pi}{2}\operatorname{cosech}\frac{s\pi}{2}}\right\}\sin\frac{s\pi y}{l} \\
\text{or} \quad \xi &= K_1 \frac{E\Phi}{2D\left(1-\frac{1}{m}\right)} \frac{8l^2}{\pi^3}
\end{aligned}\right\} \quad (10)$$

Taking the first 2 terms and assuming $m=10$ for a concrete slab,

The values of K_I in eq. (10) (see Pl. III)

($m=10$)

$z \backslash y$	0	$\frac{l}{12}$	$2\frac{l}{12}$	$3\frac{l}{12}$	$4\frac{l}{12}$	$5\frac{l}{12}$	$6\frac{l}{12}$
0	0	.3014	.5681	.7766	.9176	.9981	1.0231
$\frac{l}{12}$	0	.2997	.5649	.7720	.9119	.9917	1.0164
$2\frac{l}{12}$	0	.2945	.5546	.7574	.8938	.9712	.9953
$3\frac{l}{12}$	0	.2846	.5356	.7303	.8605	.9330	.9565
$4\frac{l}{12}$	0	.2683	.5042	.6862	.8069	.8744	.8950
$5\frac{l}{12}$	0	.2418	.4541	.6174	.7251	.7851	.8034
$6\frac{l}{12}$	0	.1977	.3729	.5104	.6038	.6574	.6742

In this case we have no large values of bending moments on the sides. It will be sufficient if we obtain the bending moments along the central lines of the plate. As for the torsion moment we have infinitely large value at the corners where we may find some difficulties to maintain the slab on the straight supports.

$$\left.\begin{aligned} M_y &= -E\Phi\frac{4}{\pi}\sum\frac{1}{s}\,\frac{\left(1-\frac{s\pi}{2}\coth\frac{s\pi}{2}\right)\cosh\frac{s\pi z}{l}+\frac{s\pi z}{l}\sinh\frac{s\pi z}{l}}{\frac{3+\frac{1}{m}}{1-\frac{1}{m}}\cosh\frac{s\pi}{2}-\frac{s\pi}{2}\operatorname{cosech}\frac{s\pi}{2}}\sin\frac{s\pi y}{l} \\ M_y &= -K_2 E\Phi \end{aligned}\right\}\cdots\cdots(11)$$

$$z=0:\quad M_y=-E\Phi\frac{4}{\pi}\sum\frac{1}{s}\,\frac{1-\frac{s\pi}{2}\coth\frac{s\pi}{2}}{\frac{3+\frac{1}{m}}{1-\frac{1}{m}}\cosh\frac{s\pi}{2}-\frac{s\pi}{2}\operatorname{cosech}\frac{s\pi}{2}}\sin\frac{s\pi y}{l}$$

$$z=\frac{l}{2}:\quad M_y=-E\Phi\frac{4}{\pi}\sum\frac{(-1)^{\frac{s-1}{2}}}{s}\,\frac{\left(1-\frac{s\pi}{2}\coth\frac{s\pi}{2}\right)\cosh\frac{s\pi z}{l}+\frac{s\pi z}{l}\sinh\frac{s\pi z}{l}}{\frac{3+\frac{1}{m}}{1-\frac{1}{m}}\cosh\frac{s\pi}{2}-\frac{s\pi}{2}\operatorname{cosech}\frac{s\pi}{2}}$$

The values of K_2 in eq. (11)

$y \backslash z$	0	$\frac{l}{12}$	$2\frac{l}{12}$	$4\frac{l}{12}$	$6\frac{l}{12}$
0	0				—
$\frac{l}{12}$	−.0357				+.241
$2\frac{l}{12}$	−.0654				+.237
$4\frac{l}{12}$	−.0984				+.214
$6\frac{l}{12}$	−.1062	−.0982	−.0715	+.0120	+.205

$$M_z=\frac{1+\frac{1}{m}}{1-\frac{1}{m}}E\Phi\frac{8}{\pi}\sum\frac{1}{s}$$

$$\left.\frac{\cosh\dfrac{s\pi z}{l}\sin\dfrac{s\pi y}{l}}{\dfrac{3+\dfrac{1}{m}}{1-\dfrac{1}{m}}\cosh\dfrac{s\pi}{2}-\dfrac{s\pi}{2}\operatorname{cosech}\dfrac{s\pi}{2}}-E\Phi-M_y\right\} \cdots(12)$$

or $\quad M_z=-K_3\,E\Phi$

The values of K_3 in eq. (12)

$y \backslash z$	0	$\frac{l}{12}$	$2\frac{l}{12}$	$4\frac{l}{12}$	$6\frac{l}{12}$
0	1.00				0
$\frac{l}{12}$	.930				0
$2\frac{l}{12}$	.864				0
$4\frac{l}{12}$	.769				0
$6\frac{l}{12}$	.720	.701	−.639	−.425	0

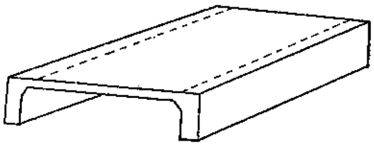


Fig. 14.

This will be applicable for a concrete slab, which is provided with 2 side stiffeners as is shown in **Fig. 14.** Some difficulties may be involved at the corners to maintain the straightness of 2 sides with stiffeners.

VII. Rectangular Plates with Three Sides Free to Rotate and One Side Fixed.

Put
$$\xi=\xi'+\xi''=\frac{E\Phi}{2D\left(1-\dfrac{1}{m}\right)}(l_1y-y^2)+\sum_s Z_s\sin\frac{s\pi y}{l_1} \quad\cdots\cdots(1)$$

For Z_s we use the full expression, as it is not, at this time, symmetrical with respect to z. (see **IV**)

$$Z_s=A_s\cosh\frac{s\pi z}{l_1}+B_s\frac{s\pi z}{l_1}\sinh\frac{s\pi z}{l_1}+C_s\sinh\frac{s\pi z}{l_1}+D_s\frac{s\pi z}{l_1}\cosh\frac{s\pi z}{l_1} \quad\cdots\cdots(2)$$

This equation must satisfy the following boundary conditions:

$$\left.\begin{array}{llll} y=0\ \&\ l_1 & : & \xi=0 \ \text{and} & M_y=0 \\ z=+\dfrac{l_2}{2} & : & \xi=0 \ \text{and} & M_z=0 \\ z=-\dfrac{l_2}{2} & : & \xi=0 \ \text{and} & \dfrac{\partial\xi}{\partial z}=0 \end{array}\right\}$$

Already we have seen that the equation (1) will satisfy the first boundary conditions. By the second conditions, we have got the following two equations:—

$$A_s \cosh \frac{s\pi l_2}{2l_1} + B_s \frac{s\pi l_2}{2l_1} \sinh \frac{s\pi l_2}{2l_1} + C_s \sinh \frac{s\pi l_2}{2l_1} + D_s \frac{s\pi l_2}{2l_1} \cosh \frac{s\pi l_2}{2l_1} = -\frac{8l_1^2}{s^3\pi^3} C \quad \ldots\ldots(3)$$

$$A_s \cosh \frac{s\pi l_2}{2l_1} + B_s \left\{ \frac{2 \cosh \frac{s\pi l}{2l_1}}{1-\frac{1}{m}} + \frac{s\pi l_2}{2l_1} \sinh \frac{s\pi l_2}{2l_1} \right\} + C_s \sinh \frac{s\pi l_2}{2l_1}$$

$$+ D_s \left\{ \frac{2 \sinh \frac{s\pi l_2}{2l_1}}{1-\frac{1}{m}} + \frac{s\pi l_2}{2l_1} \cosh \frac{s\pi l_2}{2l_1} \right\} = -\frac{8l_1^2}{s^3\pi^3} C \quad \ldots\ldots(4)$$

where $C = -\dfrac{E\Phi}{2D\left(1-\frac{1}{m}\right)}$ and $s = 1, 3, 5 \ldots\ldots$

By the third conditions we get two more equations as follows:

$$A_s \cosh \frac{s\pi l_2}{2l_1} + B_s \frac{s\pi l_2}{2l_1} \sinh \frac{s\pi l_2}{2l_1} - C_s \sinh \frac{s\pi l_2}{2l_1} - D_s \frac{s\pi l_2}{2l_1} \cosh \frac{s\pi l_2}{2l_2} = -\frac{8l_1^2}{s^3\pi^3} C \quad \ldots\ldots(5)$$

$$A_s \sinh \frac{s\pi l_2}{2l_1} + B_s \left(\sinh \frac{s\pi l_2}{2l_1} + \frac{s\pi l_2}{2l_1} \cosh \frac{s\pi l_2}{2l_1} \right) - C_s \cosh \frac{s\pi l_2}{2l_1}$$

$$- D_s \left(\cosh \frac{s\pi l_2}{2l_1} + \frac{s\pi l_2}{2l_1} \sinh \frac{s\pi l_2}{2l_1} \right) = 0 \quad \ldots\ldots(6)$$

whare C and s have the same meanings as above.

By these four equations (3), (4), (5) & (6), we can determine consts A_s, B_s, C_s, and D_s

$$A_s = -\left\{ \frac{1}{\cosh \frac{s\pi l_2}{2l_1}} + \frac{\frac{s\pi l_2}{2l_1} \tanh \frac{s\pi l_2}{2l_1}}{\cosh \frac{s\pi l_2}{2l_1} \left\{ 1 + \left(\coth \frac{s\pi l_2}{2l_1} \right)^2 \right\} - \frac{s\pi l_2}{2l_1} \frac{1}{\left(\sinh \frac{s\pi l_2}{2l_1} \right)^3}} \right\} \frac{8l_1^2}{s^3\pi^3} C$$

$$\left.\begin{aligned}
B_s &= \frac{1}{\cosh\frac{s\pi l_2}{2l_1}\left\{1+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right\}-\frac{s\pi l_2}{2l_1}\left(\operatorname{cosech}\frac{s\pi l_2}{2l_1}\right)^3}\frac{8l_1^2}{s^3\pi^3}C \\
C_s &= \frac{\frac{s\pi l_2}{2l_1}\left(\coth\frac{s\pi l_2}{2l_1}\right)^2}{\cosh\frac{s\pi l_2}{2l_1}\left\{1+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right\}-\frac{s\pi l_2}{2l_1}\left(\operatorname{cosech}\frac{s\pi l_2}{2l_1}\right)^3}\frac{8l_1^2}{s^3\pi^3}C \\
D_s &= \frac{-\coth\frac{s\pi l_2}{2l_1}}{\cosh\frac{s\pi l_2}{2l_1}\left\{1+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right\}-\frac{s\pi l_2}{2l_1}\left(\operatorname{cosech}\frac{s\pi l_2}{2l_1}\right)^3}\frac{8l_1^2}{s^3\pi^3}C
\end{aligned}\right\}\quad\ldots.(7)$$

Therefore we have

$$\begin{aligned}
\xi = \frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\frac{8l_1^2}{\pi^3}\sum_{s=1,3,..}\frac{1}{s^3}\Bigg\{1-\frac{\cosh\frac{s\pi z}{l_1}}{\cosh\frac{s\pi l_2}{2l_1}} \\
-\frac{1}{\cosh\frac{s\pi l_2}{2l_1}\left\{1+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right\}-\frac{s\pi l_2}{2l_1}\left(\operatorname{cosech}\frac{s\pi l_2}{2l_1}\right)^3}\Bigg[\Bigg(\frac{s\pi l_2}{2l_1}\tanh\frac{s\pi l_2}{2l_1} \\
+\frac{s\pi z}{l_1}\coth\frac{s\pi l_2}{2l_1}\Bigg)\cosh\frac{s\pi z}{l_1}-\left\{\frac{s\pi l_2}{2l_1}\left(\coth\frac{s\pi l_2}{2l_1}\right)^2 \right. \\
\left.+\frac{s\pi z}{l_1}\right\}\sinh\frac{s\pi z}{l_1}\Bigg]\Bigg\}\sin\frac{s\pi y}{l_1}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)
\end{aligned}$$

or,

$$\begin{aligned}
\xi = \frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\frac{8l_1^2}{\pi^3}\sum_{s=1,3...}\frac{1}{s^3} \\
\Bigg\{1-\frac{1}{\cosh\frac{s\pi l_2}{2l_1}\left\{1+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right\}-\frac{s\pi l_2}{2l_1}\left(\operatorname{cosech}\frac{s\pi l_2}{2l_1}\right)^3} \\
\Bigg[\Bigg\{1+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2+\frac{s\pi l_2}{2l_1}\left(\coth\frac{s\pi l_2}{2l_1}-\frac{\cosh\frac{s\pi l_2}{2l_1}}{\left(\sinh\frac{s\pi l_2}{2l_1}\right)^3}\right)+\frac{s\pi z}{l_1}\coth\frac{s\pi z}{l_1}\Bigg\} \\
\cosh\frac{s\pi z}{l_1}-\left\{\frac{s\pi l_2}{2l_1}\left(\coth\frac{s\pi l_2}{2l_1}\right)^2+\frac{s\pi z}{l_1}\right\}\sinh\frac{s\pi z}{l_1}\Bigg]\Bigg\}\sin\frac{s\pi y}{l_1}\quad\ldots\ldots(8)_a
\end{aligned}$$

Moments are as follows:

$$
\left.
\begin{aligned}
M_y = & -E\Phi\frac{4}{\pi}\sum_{s=1,3\ldots}\frac{1}{s} \\
& \frac{1}{\cosh\frac{s\pi l_2}{2l_1}\left\{1+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right\}-\frac{s\pi l_2}{2l_1}\left(\operatorname{cosech}\frac{s\pi l_2}{2l_1}\right)^3} \\
& \left[\left\{\frac{1+\frac{1}{m}}{1-\frac{1}{m}}+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2-\frac{s\pi l_2}{2l_1}\coth\frac{s\pi l_2}{2l_1}\left\{\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right.\right.\right. \\
& \left.\left.-2\right\}+\frac{s\pi z}{l_1}\coth\frac{s\pi l_2}{2l_1}\right\}\cosh\frac{s\pi z}{l_1}-\left\{\frac{2}{m\left(1-\frac{1}{m}\right)}\coth\frac{s\pi l_2}{2l_1}\right. \\
& \left.\left.+\frac{s\pi l_2}{2l_1}\left(\coth\frac{s\pi l_2}{2l_1}\right)^2+\frac{s\pi z}{l_1}\right\}\sinh\frac{s\pi z}{l_1}\right]\sin\frac{s\pi y}{l_1} \\
M_z = & -E\Phi\left[1+\frac{4}{\pi}\sum_{s=1,3\ldots}\frac{1}{s}\right. \\
& \frac{1}{\cosh\frac{s\pi l_2}{2l_1}\left\{1+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right\}-\frac{s\pi l_2}{2l_1}\left(\operatorname{cosech}\frac{s\pi l_2}{2l_1}\right)^3} \\
& \left[\left\{\frac{1+\frac{1}{m}}{1-\frac{1}{m}}-\left(\coth\frac{s\pi l_2}{2l_1}\right)^2+\frac{s\pi l_2}{2l_1}\coth\frac{s\pi l_2}{2l_1}\left\{\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right.\right.\right. \\
& \left.\left.-2\right\}-\frac{s\pi z}{l_1}\coth\frac{s\pi l_2}{2l_1}\right\}\cosh\frac{s\pi z}{l_1}-\left\{\frac{2}{1-\frac{1}{m}}\coth\frac{s\pi l_2}{2l_1}\right. \\
& \left.\left.\left.-\frac{s\pi l_2}{2l_1}\left(\coth\frac{s\pi l_2}{2l_1}\right)^2-\frac{s\pi z}{l_1}\right\}\sinh\frac{s\pi z}{l_1}\right]\sinh\frac{s\pi y}{l_1}\right] \\
M_{yz} = & -E\Phi\frac{4}{\pi}\sum\frac{1}{s}
\end{aligned}
\right\}\quad\ldots(9)
$$

$$\frac{1}{\cosh\frac{s\pi l_2}{2l_1}\left\{1+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right\}-\frac{s\pi l_2}{2l_1}\left(\mathrm{cosech}\,\frac{s\pi l_2}{2l_1}\right)^3}$$
$$\left[\left\{-\left(\coth\frac{s\pi l_2}{2l_1}\right)^2+\frac{s\pi l_2}{2l_1}\coth\frac{s\pi l_2}{2l_1}\left\{\left(\coth\frac{s\pi l_2}{2l_1}\right)^2-2\right\}\right.\right.$$
$$\left.-\frac{s\pi z}{l_1}\coth\frac{s\pi l_2}{2l_1}\right\}\sinh\frac{s\pi z}{l_1}-\left\{\coth\frac{s\pi l_2}{2l_1}-\frac{s\pi l_2}{2l_1}\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right.$$
$$\left.\left.-\frac{s\pi z}{l_1}\right\}\cosh\frac{s\pi z}{l_1}\right]\cos\frac{s\pi y}{l_1}\Bigg)$$

As to Eulerian Mean Curvature:—

$$\Delta\xi=-\frac{E\Phi}{\left(1-\frac{1}{m}\right)D}-\frac{E\Phi}{\left(1-\frac{1}{m}\right)D}\frac{4}{\pi}\sum\frac{1}{s}$$
$$\frac{1}{\cosh\frac{s\pi l_2}{2l_1}\left\{1+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right\}-\frac{s\pi l_2}{2l_1}\left(\mathrm{cosech}\,\frac{s\pi l_2}{2l_1}\right)^3}\left[-2\cosh\frac{s\pi z}{l_1}\right.$$
$$\left.+2\coth\frac{s\pi l_2}{2l_1}\sinh\frac{s\pi z}{l_1}\right]\sin\frac{s\pi y}{l_1}$$

The first term of the right hand member is equal to the case of 4 sides free to rotate. The second term converges quickly. This relation leads us to obtain the value of M_z from the value of M_y;

$$M_z=-M_y-E\Phi+\frac{1+\frac{1}{m}}{1-\frac{1}{m}}E\Phi\frac{4}{\pi}\sum\frac{1}{s}$$
$$\frac{2}{\cosh\frac{s\pi l_2}{2l_1}\left\{1+\left(\coth\frac{s\pi l_2}{2l_1}\right)^2\right\}-\frac{s\pi l_2}{2l_1}\left(\mathrm{cosech}\,\frac{s\pi l_2}{2l_1}\right)^3}\left[-\cosh\frac{s\pi z}{l_1}\right.$$
$$\left.+\coth\frac{s\pi l_2}{2l_1}\sinh\frac{s\pi z}{l_1}\right]\sin\frac{s\pi y}{l_1}$$

We obtain $M_y=-E\Phi$ and $M_z=0$ at $z=\frac{l_2}{2}$, which are of the same values as in the case of 4 sides free to rotate. The expression for shears and reactions are omitted.

Numerical Example: (a square plate) (see **Pl. IV.**)

Put $l_1=l_2=l$

$$\begin{aligned}\xi=&\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\frac{8l_1^2}{\pi^3}\sum\frac{1}{s^3}\\
&\left\{1-\frac{1}{\cosh\frac{s\pi}{2}\left\{1+\left(\coth\frac{s\pi}{2}\right)^2\right\}-\frac{s\pi}{2}\left(\operatorname{cosech}\frac{s\pi}{2}\right)^3}\right.\\
&\left[\left\{\left\{1+\left(\coth\frac{s\pi}{2}\right)^2+\frac{s\pi}{2}\left\{\coth\frac{s\pi}{2}-\cosh\frac{s\pi}{2}\left(\operatorname{cosech}\frac{s\pi}{2}\right)^3\right\}\right.\right.\right.\\
&\left.+\frac{s\pi z}{l}\coth\frac{s\pi}{2}\right\}\cosh\frac{s\pi z}{l}-\left\{\frac{s\pi}{2}\left(\coth\frac{s\pi}{2}\right)^2\right.\\
&\left.\left.\left.+\frac{s\pi z}{l}\right\}\sinh\frac{s\pi z}{l}\right]\right\}\Bigg]\sin\frac{s\pi y}{l}\qquad\cdots\cdots\cdots\cdots(10)\end{aligned}$$

or
$$\xi=K_1\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)}\frac{8l^2}{\pi^3}$$

Taking the first 2 terms,

The values of K_1 in eq, (10)

z \ y	0	$\frac{l}{12}$	$2\frac{l}{12}$	$3\frac{l}{12}$	$4\frac{l}{12}$	$5\frac{l}{12}$	$6\frac{l}{12}$
0	0	.111	.201	.260	.288	.296	.298
$\frac{l}{12}$	0	.118	.215	.279	.311	.322	.325
$2\frac{l}{12}$	0	.119	.216	.281	.313	.325	.327
$3\frac{l}{12}$	0	.110	.200	.260	.289	.299	.301
$4\frac{l}{12}$	0	.091	.164	.212	.234	.241	.242
$5\frac{l}{12}$	0	.056	.101	.129	.141	.143	.143
$6\frac{l}{12}$	0	0	0	0	0	0	0
$-\frac{l}{12}$	0	.098	.176	.226	.247	.252	.252
$-2\frac{l}{12}$	0	.079	.142	.179	.193	.195	.193
$-3\frac{l}{12}$	0	.058	.101	.126	.132	.129	.128
$-4\frac{l}{12}$	0	.033	.058	.070	.071	.067	.065
$-5\frac{l}{12}$	0	.012	.020	.023	.022	.020	.019
$-6\frac{l}{12}$	0	0	0	0	0	0	0

The values of M_y are as follows:

$$M_y = -E\Phi\frac{4}{\pi}\sum\frac{1}{s}\frac{1}{\cosh\frac{s\pi}{2}\left\{1+\left(\coth\frac{s\pi}{2}\right)^2\right\}-\frac{s\pi}{2}\left(\operatorname{cosech}\frac{s\pi}{2}\right)^3}$$
$$\left[\left\{\frac{1+\frac{1}{m}}{1-\frac{1}{m}}+\left(\coth\frac{s\pi}{2}\right)^2-\frac{s\pi}{2}\coth\frac{s\pi}{2}\left\{\left(\coth\frac{s\pi}{2}\right)^2\right.\right.\right.$$
$$\left.\left.-2\right\}+\frac{s\pi z}{2}\coth\frac{s\pi}{2}\right\}\cosh\frac{s\pi z}{l}-\left\{\frac{2}{m\left(1-\frac{1}{m}\right)}\coth\frac{s\pi}{2}\right.$$
$$\left.\left.+\frac{s\pi}{2}\left(\coth\frac{s\pi}{2}\right)^2+\frac{s\pi z}{l}\right\}\sinh\frac{s\pi z}{l}\right]\sin\frac{s\pi y}{l} \quad \ldots\ldots(11)$$

or $\qquad M_y = -K_2E\Phi$

The values along the central lines are given in the following table.

The values of K_2 in eq. (11)

$(m=10)$

y \ z	0	$2\frac{l}{12}$	$4\frac{l}{12}$	$6\frac{l}{12}$	$-2\frac{l}{12}$	$-4\frac{l}{12}$	$-6\frac{l}{12}$
0	0						
$\frac{l}{12}$	.2533						
$2\frac{l}{12}$	.4526						
$4\frac{l}{12}$	.7803						
$6\frac{l}{12}$	.8767	.818	.866	1.00	1.007	1.131	1.139

The values of M_z are given by the following equation;

$$M_z = -M_y - E\Phi + \frac{1+\frac{1}{m}}{1-\frac{1}{m}}E\Phi\frac{4}{\pi}\sum\frac{1}{s}$$
$$\frac{2}{\cosh\frac{s\pi}{2}\left\{1+\left(\coth\frac{s\pi}{2}\right)^2\right\}-\frac{s\pi}{2}\left(\operatorname{cosech}\frac{s\pi}{2}\right)^3} \quad \ldots\ldots(12)$$

$$\left\{-\cosh\frac{s\pi z}{l}+\coth\frac{s\pi}{2}\sinh\frac{s\pi z}{l}\right\}\sin\frac{s\pi y}{l}$$

or $$M_z=-K_3\,E\Phi$$

The values along the central lines are given in the following table.

The values of K_3 in eq. (12)

y \ z	0	$2\frac{l}{12}$	$4\frac{l}{12}$	$6\frac{l}{12}$	$-2\frac{l}{12}$	$-4\frac{l}{12}$	$-6\frac{l}{12}$
$\frac{l}{12}$	.904						
$2\frac{l}{12}$	.847						
$4\frac{l}{12}$	.722						
$6\frac{l}{12}$	.694	.495	.272	0	.960	1.407	1.978

Remarks: In the same way, we can obtain the expressions for two more asymmetrical cases, i. e., the one case is with three sides free to rotate and with one side unsupported, while the other case is with two sides free to rotate and with one side fixed and the remaining side unsupported. And, also, the expressions for the semi-infinitely extended strip-plates with various kinds of supports are easily obtained, though we will not give their expressions here.

VIII. Circular Plates.

We will change the cartesian co-ordinates y and z of the fundamental equations of thermal flexure into the polar co-ordinates by the following relations.

$$y=r\cos\varphi$$
$$z=r\sin\varphi$$

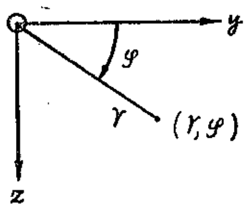


Fig. 15.

By the usual change of variables, we have

$$\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}\cos^2\varphi+\frac{1}{r^2}\sin^2\varphi\frac{\partial^2}{\partial\varphi^2}+\frac{1}{r^2}\sin^2\varphi\frac{\partial}{\partial r}-\frac{2}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\,\partial\varphi}+\frac{2}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}$$

$$\frac{\partial^2}{\partial z^2}=\sin^2\varphi\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}+\frac{1}{r}\cos^2\varphi\frac{\partial}{\partial r}+\frac{2}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\,\partial\varphi}-\frac{2}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial y\,\partial z}=\sin\varphi\cos\varphi\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial^2}{\partial\varphi^2}+\frac{1}{r}(\cos^2\varphi-\sin^2\varphi)\frac{\partial}{\partial r\,\partial\varphi}$$
$$-\frac{1}{r}\sin\varphi\cos\varphi\frac{\partial}{\partial r}+\frac{1}{r^2}(\sin^2\varphi-\cos^2\varphi)\frac{\partial}{\partial\varphi}$$

If we make the radius vector coincide with y-axis, i. e., put $\varphi=0$, we have

$$\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}$$
$$\frac{\partial^2}{\partial z^2}=\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$
$$\frac{\partial}{\partial y\,\partial z}=\frac{1}{r}\frac{\partial^2}{\partial r\,\partial\varphi}-\frac{1}{r^2}\frac{\partial}{\partial\varphi}$$

Therefore we obtain the following expressions for stresses and moments directly from the equations in **II**;—

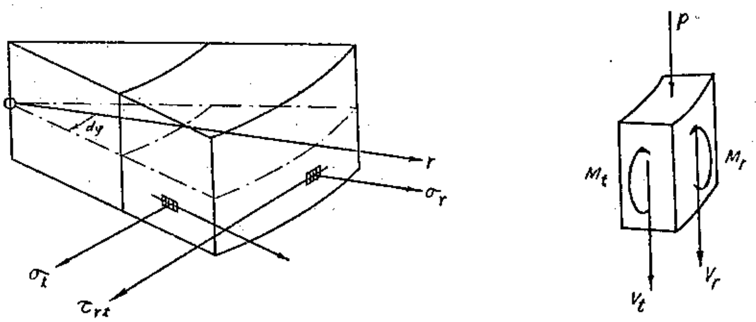


Fig. 16.

$$\left.\begin{aligned}
\sigma_r&=-\frac{E}{1-\frac{1}{m^2}}\left\{x\frac{\partial^2\xi}{\partial r^2}+\frac{x}{m}\left(\frac{1}{r}\frac{\partial\xi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\xi}{\partial\varphi^2}\right)\right.\\
&\qquad\left.+\alpha\left(1+\frac{1}{m}\right)(\theta-\theta_m)\right\}\\
\sigma_t&=-\frac{E}{1-\frac{1}{m^2}}\left\{x\left(\frac{1}{r}\frac{\partial\xi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\xi}{\partial\varphi^2}\right)+\frac{x}{m}\frac{\partial^2\xi}{\partial r^2}\right.\\
&\qquad\left.+\alpha\left(1+\frac{1}{m}\right)(\theta-\theta_m)\right\}\\
\sigma_{rt}&=-2xG\left(\frac{1}{r}\frac{\partial^2\xi}{\partial r\,\partial\varphi}-\frac{1}{r^2}\frac{\partial\xi}{\partial\varphi}\right)
\end{aligned}\right\}\quad\ldots.(1)$$

where suffixes r and t mean radial and tangential directions resp. (**Fig. 16**)

$$\left.\begin{aligned}
M_r &= -D\left\{\frac{\partial^2\xi}{\partial r^2}+\frac{1}{m}\left(\frac{1}{r}\frac{\partial\xi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\xi}{\partial\varphi^2}\right)\right\}-\frac{E\Phi}{1-\frac{1}{m}} \\
M_t &= -D\left\{\frac{1}{r}\frac{\partial\xi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\xi}{\partial\varphi^2}+\frac{1}{m}\frac{\partial^2\xi}{\partial r^2}\right\}-\frac{E\Phi}{1-\frac{1}{m}} \\
M_{rt} &= -\left(1-\frac{1}{m}\right)D\left(\frac{1}{r}\frac{\partial^2\xi}{\partial r\,\partial\varphi}-\frac{1}{r^2}\frac{\partial\xi}{\partial\varphi}\right)
\end{aligned}\right\} \quad \ldots\ldots(2)$$

where $D=\dfrac{EI}{1-\frac{1}{m^2}}$ and $\Phi=\int\alpha\,(\theta-\theta_m)\,x\,dx$

Shears and reactions are as follows:

$$\left.\begin{aligned}
V_r &= -D\frac{\partial}{\partial r}\Delta\xi \\
V_t &= -D\frac{\partial}{\partial r\,\partial\varphi}\Delta\xi
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $\Delta=\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2}{\partial\varphi^2}$

$$\left.\begin{aligned}
R_r &= \left[V_r+\frac{\partial M_{rt}}{r\,\partial\varphi}\right]_{r=r_0} \\
R_t &= \left[V_t+\frac{\partial M_{rt}}{\partial r}\right]_{\varphi=\varphi_0}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(4)$$

The condition of equilibrium is, p being the distributed loads on the plate,

$$\Delta\Delta\xi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)^2\xi=\frac{p}{D} \quad \ldots\ldots\ldots(5)$$

If we consider only the temperature bending, $p=0$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)^2\xi=0 \quad \ldots\ldots\ldots\ldots\ldots(5)_a$$

When the values of ξ have an axial symmetry, losing the terms about φ, the above equations are simplified into the following forms:

$$\left.\begin{aligned}
\sigma_r &= -\frac{E}{1-\frac{1}{m^2}}\left\{x\frac{d^2\xi}{dr^2}+\frac{1}{m}\frac{x}{r}\frac{d\xi}{dr}+\alpha\left(1+\frac{1}{m}\right)(\theta-\theta_m)\right\} \\
\sigma_t &= -\frac{E}{1-\frac{1}{m^2}}\left\{\frac{x}{r}\frac{d\xi}{dr}+\frac{x}{m}\frac{d^2\xi}{dr^2}+\alpha\left(1+\frac{1}{m}\right)(\theta-\theta_m)\right\} \\
\tau_{rt} &= 0
\end{aligned}\right\} \quad \ldots\ldots(6)$$

$$\left.\begin{aligned} M_r &= -D\left\{\frac{d^2\xi}{dr^2}+\frac{1}{m}\frac{1}{r}\frac{d\xi}{dr}\right\}-\frac{E\Phi}{1-\frac{1}{m}} \\ M_t &= -D\left\{\frac{1}{r}\frac{d\xi}{dr}+\frac{1}{m}\frac{d^2\xi}{dr^2}\right\}-\frac{E\Phi}{1-\frac{1}{m}} \\ M_{rt} &= 0 \end{aligned}\right\} \quad \cdots\cdots\cdots(7)$$

$$\left.\begin{aligned} V_r &= -D\frac{d}{dr}\Delta\xi \\ V_t &= 0, \qquad \Delta \equiv \frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr} \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(8)$$

$$\left.\begin{aligned} R_r &= [V_r]_{r=r_0} \\ R_t &= 0 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(9)$$

$$\Delta\Delta\xi=\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)^2\xi=0 \quad \cdots\cdots\cdots\cdots\cdots(10)$$

The general solution of equation (10) is well known in the following form:

$$\xi=C_1+C_2r^2+C_3r^2\log r+C_4\log r \quad \cdots\cdots\cdots\cdots\cdots(11)$$

If a circular plate is supported so as the deflection angles at every points on its boundary to be equal to zero and to yield freely in radial directions, we have $\xi=0$ throughout the plate. We have $C_3=0$ and $C_4=0$ from the conditions that ξ and M_r will not tend to infinity at the origin, and $C_1=C_2=0$ from the conditions that ξ together with $\frac{d\xi}{dr}$ is equal to zero on the boundary. In this case we have $M_r=M_t=-\frac{E\Phi}{1-\frac{1}{m}}$ as it is obvious in the fundamental equations.

If a circular plate is supported free to rotate and is to yield in the radial directions, we have $C_3=C_4=0$ just as in the above case. And $\xi=0$ & $M_r=0$ on the boundary, $r=r_0$, give:

$$\xi=0: \quad C_1+C_2r_0^2=0$$

$$M_r=0: \quad -D\left\{2C_2+\frac{2C_2}{m}\right\}-\frac{E\Phi}{1-\frac{1}{m}}=0$$

$$\therefore \qquad C_1=\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)\left(1+\frac{1}{m}\right)}r_0^2$$

$$C_2 = -\frac{E\Phi}{2D\left(1-\frac{1}{m}\right)\left(1+\frac{1}{m}\right)}$$

Therefore we have,

$$\xi = \frac{E\Phi}{2D\left(1-\frac{1}{m^2}\right)}\{r_0^2 - r^2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

This is a paraboloid of revolution or a spherical surface, if r is small. In this case, at any point of the plate, we have,

$$M_r = 0, \quad M_t = 0, \quad V_r = 0 \quad \text{and} \quad R_r = 0$$

i. e., it is *the same with the case of the boundary not supported at all.* Next, we deal with *the case of a ring-shape plate with boundaries free to rotate,*

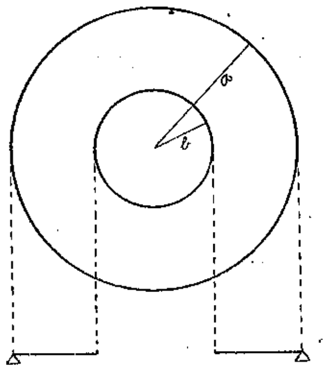


Fig. 17

We have the following boundary conditions to determine constants in equation (11)

$$\left.\begin{array}{llll} \text{at} & r=a: & \xi=0 & \text{and} \quad M_r=0 \\ \text{at} & r=b: & M_r=0 & \text{and} \quad R_r=0 \end{array}\right\}$$

These conditions will give the following equations:—

$$\left.\begin{aligned}
&C_1 + C_2 a^2 + C_3 a^2 \log a + C_4 \log a = 0 \\
&2\left(1+\frac{1}{m}\right)C_2 + 2C_3\left(1+\frac{1}{m}\right)\log a + \left(3+\frac{1}{m}\right)C_3 - \left(1-\frac{1}{m}\right)\frac{C_4}{a^2} = -\frac{E\Phi}{D\left(1-\frac{1}{m}\right)} \\
&2\left(1+\frac{1}{m}\right)C_2 + 2C_3\left(1+\frac{1}{m}\right)\log b + \left(3+\frac{1}{m}\right)C_3 - \left(1-\frac{1}{m}\right)\frac{C_4}{b^2} = -\frac{E\Phi}{D\left(1-\frac{1}{m}\right)} \\
&-D\frac{4C_3}{b} = 0
\end{aligned}\right\}$$

We have

$$\left.\begin{aligned}
C_1 &= \frac{E\Phi a^2}{2D\left(1-\frac{1}{m^2}\right)} \\
C_2 &= -\frac{E\Phi}{2D\left(1-\frac{1}{m^2}\right)} \\
C_3 &= C_4 = 0
\end{aligned}\right\}$$

Or, we have just the same equation as (12)

$$\xi = \frac{E\Phi}{2D\left(1-\frac{1}{m^2}\right)}\{a^2-r^2\} \quad \ldots\ldots\ldots\ldots\ldots (13)$$

If it is fixed at the external boundary so that it may have zero deflection angle and may yield only radially on it; The boundary conditions are,

$$\left.\begin{array}{llll} r=a: & \xi=0 & \text{and} & \dfrac{d\xi}{dr}=0 \\ r=b: & M_r=0 & \text{and} & R_r=0 \end{array}\right\}$$

These conditions will give $C_3=0$ as in the preceeding case.

For the determination of other constants we have the equations of conditions,

$$\left.\begin{array}{l} C_1+C_2a^2+C_4\log a=0 \\ 2C_2a+\dfrac{C_4}{a}=0 \\ 2\left(1+\dfrac{1}{m}\right)C_2-\left(1-\dfrac{1}{m}\right)\dfrac{C_4}{b^2}=-\dfrac{E\Phi}{D\left(1-\frac{1}{m}\right)} \end{array}\right\}$$

which give:

$$\left.\begin{array}{l} C_1=\dfrac{E\Phi\left(\frac{1}{2}-\log a\right)}{\left(1-\frac{1}{m}\right)D\left\{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{m}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\right\}} \\ C_2=-\dfrac{E\Phi\frac{1}{2a^2}}{\left(1-\frac{1}{m}\right)D\left\{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{m}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\right\}} \\ C_4=\dfrac{E\Phi}{\left(1-\frac{1}{m}\right)D\left\{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{m}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\right\}} \end{array}\right\}$$

Therefore we have,

$$\xi=\frac{E\Phi\left\{\frac{1}{2}\left(1-\frac{r^2}{a^2}\right)+\log\frac{r}{a}\right\}}{D\left(1-\frac{1}{m}\right)\left\{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{m}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\right\}} \quad \ldots\ldots\ldots (14)$$

The moments are,

$$M_r=-\frac{E\Phi}{1-\frac{1}{m}}\left\{1-\frac{\frac{1}{a^2}+\frac{1}{r^2}+\frac{1}{m}\left(\frac{1}{a^2}-\frac{1}{r^2}\right)}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{m}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}\right\}$$

$$M_t=-\frac{E\Phi}{1-\frac{1}{m}}\left\{1-\frac{\frac{1}{a^2}-\frac{1}{r^2}+\frac{1}{m}\left(\frac{1}{a^2}+\frac{1}{r^2}\right)}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{m}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}\right\} \quad \ldots\ldots(15)$$

$$M_{rt}=0$$

at $r=a$:

$$M_r=-\frac{E\Phi}{1-\frac{1}{m}}\left\{1-\frac{\frac{2}{a^2}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{m}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}\right\}$$

$$M_t=-\frac{E\Phi}{1-\frac{1}{m}}\left\{1-\frac{\frac{1}{m}\frac{2}{a^2}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{m}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}\right\}$$

at $r=b$: $M_r=0$

$$M_t=-\frac{E\Phi}{1-\frac{1}{m}}\,\frac{\frac{2}{b^2}\left(1-\frac{1}{m}\right)}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{m}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}$$

As to Eulerian Mean Curvature:—

$$\Delta\xi=-\frac{E\Phi}{D\left(1-\frac{1}{m}\right)}\,\frac{\frac{2}{a^2}}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{m}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)},$$

constant throughout.

Therefore, we have $V_r=0$, throughout. V_t is, of course, equal to zero.

Numerical example: (see **Pl. V**)

Take a case $a=2b$ and $m=10$.

$$\xi=\frac{E\Phi}{\left(1-\frac{1}{m}\right)D}\,\frac{40}{47}\left\{\frac{1}{2}\left(1-\frac{r^2}{4b^2}\right)+\log\frac{r}{2b}\right\}$$

or

$$\xi=K_1\frac{E\Phi}{\left(1-\frac{1}{m}\right)D} \quad \ldots\ldots(16)$$

The values of K_1 in eq. (16)

($m=10$)

r	b	$\frac{3}{2}b$	$2b$
$A\equiv\frac{1}{2}\left(1-\frac{r_2}{4b^2}\right)$	.375	.281	0

$B = \log \frac{r}{2b}$	$-.693$	$-.288$	
$\frac{40}{47}(A+B)$	$-.271$	$-.006$	0

$$\left.\begin{aligned} M_r &= -\frac{E\Phi}{1-\frac{1}{m}}\,\frac{36}{47}\left(1-\frac{b^2}{r^2}\right) \quad \text{or} \quad M_r = -K_2\frac{E\Phi}{1-\frac{1}{m}} \\ M_t &= -\frac{E\Phi}{1-\frac{1}{m}}\,\frac{36}{47}\left(1-\frac{b^2}{r^2}\right) \quad \text{or} \quad M_t = -K_3\frac{E\Phi}{1-\frac{1}{m}} \end{aligned}\right\} \quad \ldots(17)$$

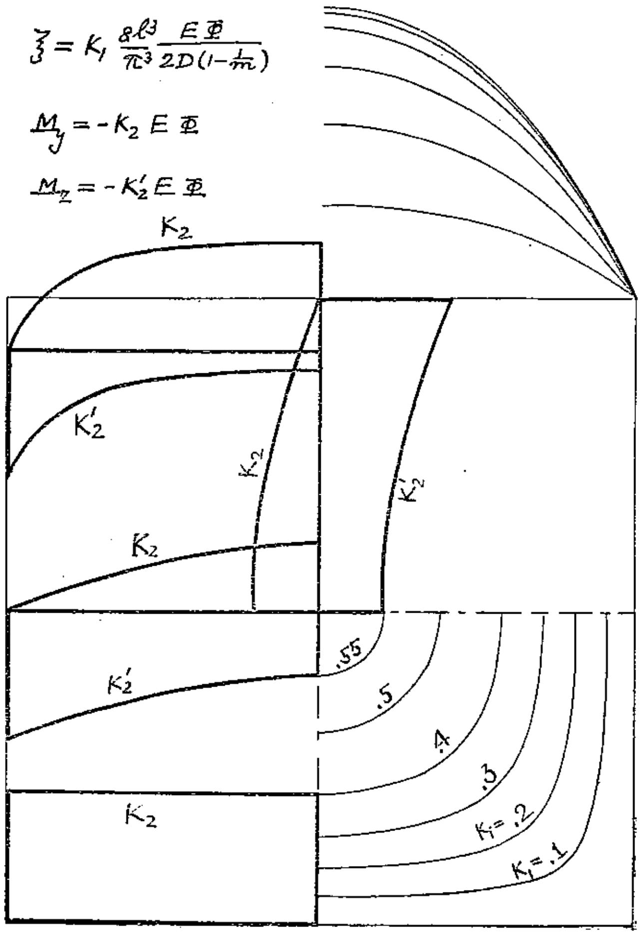
The values of K_2 and K_3 in eg. (17)

r	b	$\frac{3}{2}b$	$2b$
K_2	0	.426	.575
K_3	1.532	1.106	.957

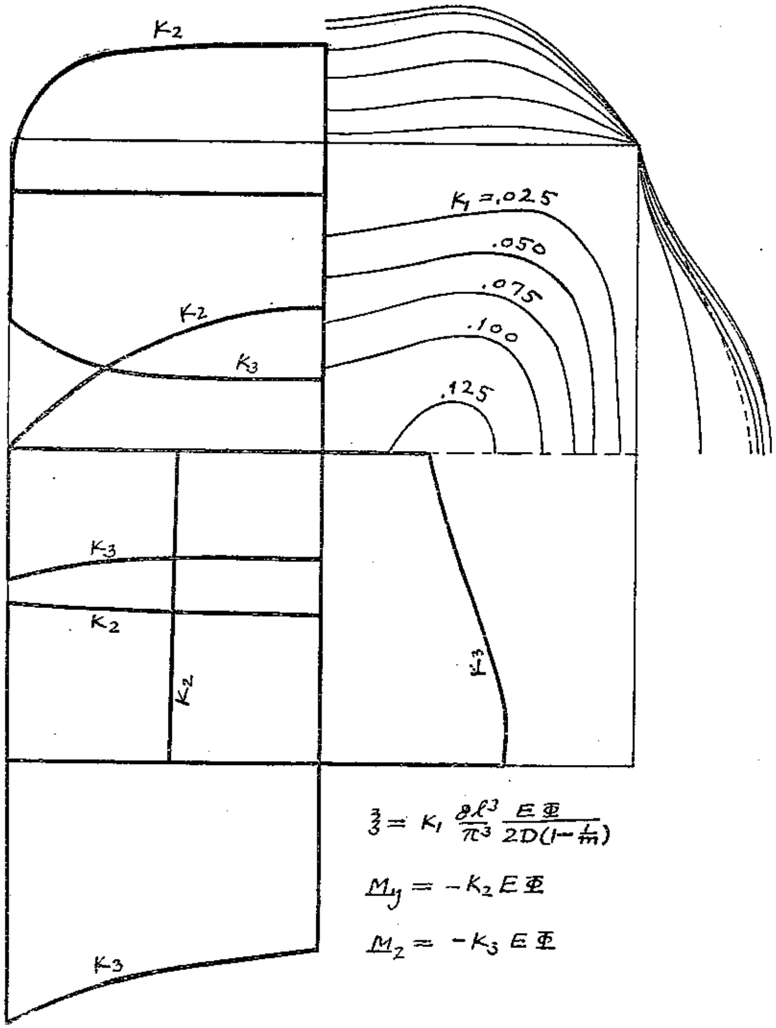
The end.

Tokyo, March 1927.

Pl. I

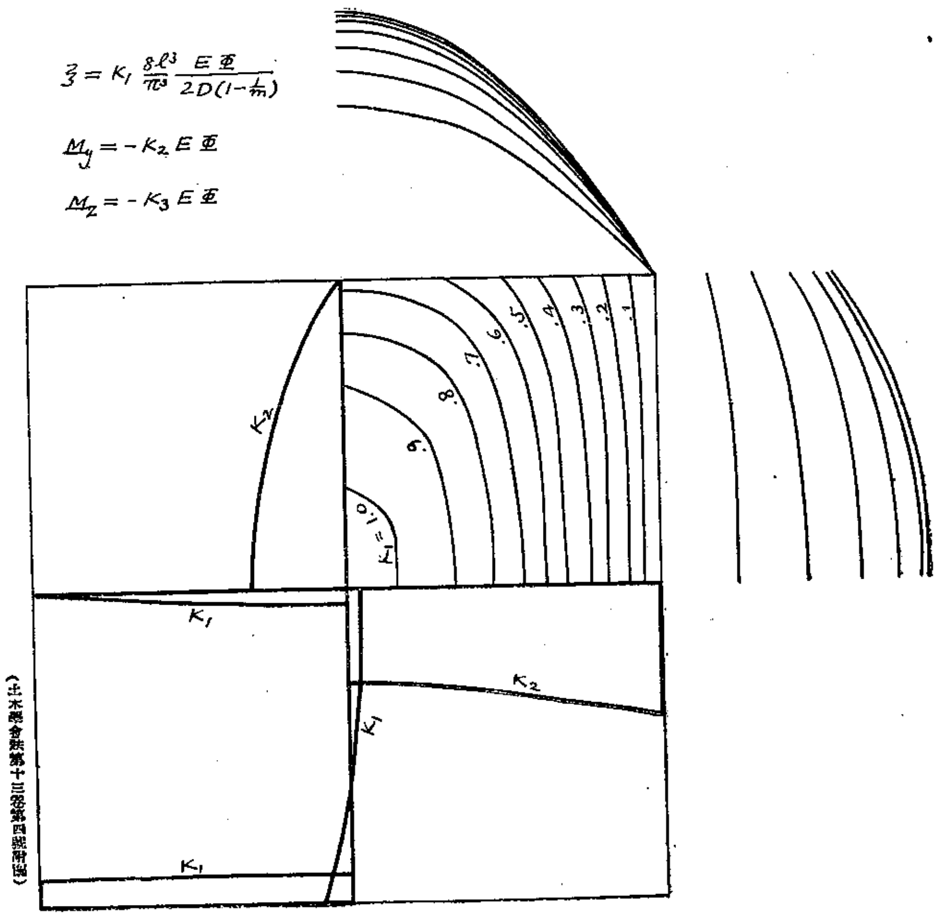


Pl. II



(土木學會誌第十三卷第四號附圖)

Pl. III



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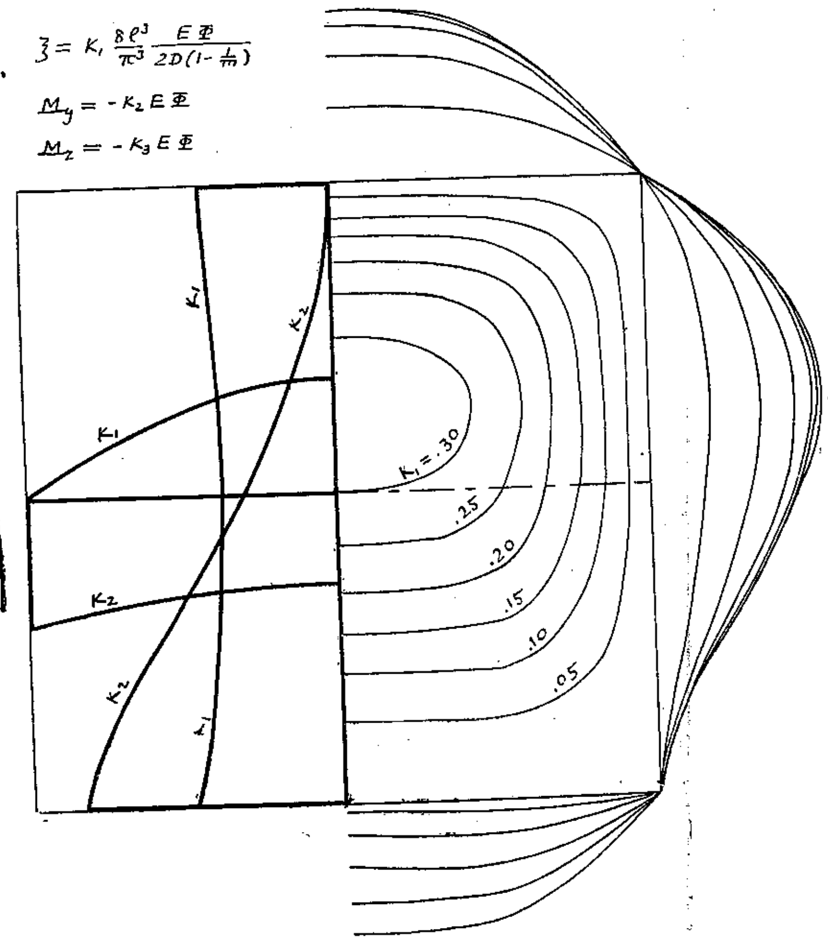
Pl. IV



Pl. V