Consideration on Covered Reservoir Wall acting as a Beam subjected to a Load of Uniformly Varying Intensity.

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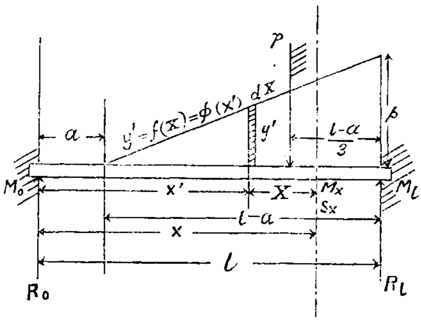
— I. GENERAL CASE —

A). as Beam both ends fixed: —

Let the section of reservoir wall under consideration be shown by the following figure, in which the wall is drawn horizontal and the acting forces vertical for the sake of convenience in discussion.



Referring to the fig., let

l = span of the beam, or total height of the reservoir wall;

a = length of the unloaded portion of the beam;

P = resultant load due to water or earth pressure according to the condition;

p = max. intensity of the load;

x = distance from the left end to any point in the beam;

M_x = bending moment at the section x;

S_x = vertical shear at the section x;

R = reaction at end of the beam;

y' = intensity of the load at any distance x' in the beam from the left end, or distance X from the section x toward the left end;

then we have

$$P=\frac{p(l-a)}{2} \quad \text{and hence} \quad p=\frac{2P}{(l-a)}.$$

Now, the equation to intensity of the load with origin at the left end may be expressed

$$y'=f(X)=\phi(x')=m(x'-a),$$

in which $m=tan\ a=\dfrac{p}{l-a}=\dfrac{2P}{(l-a)^2}$,

so that $y'=\dfrac{2P}{(l-a)^2}(x'-a)$, where $x' \nless a$ ....(1)

Hence the bending moment at any point in the beam may be written in the following forms according as $x \ngtr a$ or $x \nless a$:

$$\begin{cases} M_{x\ngtr a}=M_0+R_0x \\ \text{or } M_{x\nless a}=M_0+R_0x-\displaystyle\int_a^x y'dX(-X), \end{cases}$$

where M_0 & R_0 denote the bending moment and reaction at the left end.

Since $X=x-x'$ and also $d\ X=-dx'$, we have

$$\int_a^x y'dX(-X)=\int_a^x y'd\,x'(x-x')=\int_a^x \frac{2P}{(l-a)^2}(x'-a)(x-x')dx'$$

$$=\int_a^x \frac{2P}{(l-a)^2}(xx'-ax-x'^2+ax')dx'$$

$$=\frac{2P(x+a)}{(l-a)^2}\int_a^x x'dx'-\frac{2P}{(l-a)^2}\int_a^x x'^2dx'-\frac{2Pax}{(l-a)^2}\int_a^x dx';$$

and performing integrations,

$$\frac{2P(x+a)}{(l-a)^2}\left[\frac{x'^2}{2}\right]_a^x-\frac{2P}{(l-a)^2}\left[\frac{x'^3}{3}\right]_a^x-\frac{2Pax}{(l-a)^2}\left[x'\right]_a^x$$

$$=\frac{2P(x+a)(x^2-a^2)}{2(l-a)^2}-\frac{2P(x^3-a^3)}{3(l-a)^2}-\frac{2Pax(x-a)}{(l-a)^2}=\frac{P(x-a)^3}{3(l-a)^2}$$

Hence

$$M_{x \ngtr a}=M_0+R_0x \quad\cdots\cdots (2)$$

or

$$M_{x \nless a}=M_0+R_0x-\frac{P(x-a)^3}{3(l-a)^2} \quad\cdots\cdots (3)$$

Since the equations of bending moment are different on the loaded portion and on the unloaded portion of the beam, there are two elastic curves which have a common ordinate and a common tangent at the point where $x=a$.

Let y = ordinate of the elastic curve of the beam at the point where the abscissa is x, with origin at the left end;

E = Modulus of Elasticity of the material of beam;

I = Moment of inertia of the cross section of beam;

then from the general differential equation of elastic curve $EI\frac{d^2y}{dx^2}=M$, assuming as usual E & I to be constant, we have in the present case,

for $x \ngtr a$

$$EI\left(\frac{d^2y}{dx^2}\right)_{x\ngtr a}=M_0+R_0x \quad\cdots\cdots (4)$$

Integrating once, $EI\left(\frac{dy}{dx}\right)_{x\ngtr a}=M_0x+\frac{R_0x^2}{2}+c_1$,

for $x=0$, $\frac{dy}{dx}=0$ and hence $c_1=0$,

so that

$$EI\left(\frac{dy}{dx}\right)_{x\ngtr a}=M_0x+\frac{R_0x^2}{2} \quad\cdots\cdots (5)$$

Integrating again, $EI\,(y)_{x\ngtr a}=\frac{M_0x^2}{2}+\frac{R_0x^3}{6}+c_2$, for $x=0$, $y=0$ and hence $c_2=0$

so that $$\mathrm{EI}(y)_{x>a}=\frac{M_0x^2}{2}+\frac{R_0x^3}{6} \quad \cdots\cdots(6)$$

Also for $x \not< a$

$$\mathrm{EI}\left(\frac{d^2y}{dx^2}\right)_{x\not<a}=M_0+R_0x-\frac{P}{3(l-a)^2}(x^3-3x^2a+3xa^2-a^3) \quad \cdots\cdots(7)$$

Integrating once,

$$\mathrm{EI}\left(\frac{dy}{dx}\right)_{x\not<a}=M_0x+\frac{R_0x^2}{2}-\frac{P}{3(l-a)^2}\left(\frac{x^4}{4}-\frac{3ax^3}{3}+\frac{3a^2x^2}{2}-a^3x\right)+c_3$$

for $x=a$, $\left(\frac{dy}{dx}\right)_{x>a}=\left(\frac{dy}{dx}\right)_{x\not<a}$, and hence $c_3=-\frac{Pa^4}{12(l-a)^2}$,

so that $$\mathrm{EI}\left(\frac{dy}{dx}\right)_{x\not<a}=M_0x+\frac{R_0x^2}{2}-\frac{P}{3(l-a)^2}\left(\frac{x^4}{4}-ax^3+\frac{3a^2x^2}{2}-a^3x+\frac{a^4}{4}\right) \cdots\cdots(8)$$

Integrating again,

$$\mathrm{EI}(y)_{x\not<a}=\frac{M_0x^2}{2}+\frac{R_0x^3}{6}-\frac{P}{3(l-a)^2}\left(\frac{x^5}{20}-\frac{ax^4}{4}+\frac{3a^2x^3}{6}-\frac{a^3x^2}{2}+\frac{a^4x}{4}\right)+c_4$$

for $x=a$, $(y)_{x>a}=(y)_{x\not<a}$, and heace $c_4=\frac{Pa^5}{60(l-a)^2}$

so that

$$\mathrm{EI}(y)_{x\not<a}=\frac{M_0x^2}{2}+\frac{R_0x^3}{6}-\frac{P}{3(l-a)^2}\left(\frac{x^5}{20}-\frac{ax^4}{4}+\frac{a^2x^3}{2}-\frac{a^3x^2}{2}+\frac{a^4x}{4}-\frac{a^5}{20}\right) \cdots\cdots(9)$$

Again for $x=l$, we have $\frac{dy}{dx}=0$ and also $y=0$; hence from (8) and (9)

$$\begin{cases} M_0l+\dfrac{R_0l^2}{2}-\dfrac{P}{3(l-a)^2}\left(\dfrac{l^4}{4}-al^3+\dfrac{3a^2l^2}{2}-a^3l+\dfrac{a^4}{4}\right)=0 \\ \dfrac{M_0l^2}{2}+\dfrac{R_0l^3}{6}-\dfrac{P}{3(l-a)^2}\left(\dfrac{l^5}{20}-\dfrac{al^4}{4}+\dfrac{a^2l^3}{2}-\dfrac{a^3l^2}{2}+\dfrac{a^4l}{4}-\dfrac{a^5}{20}\right)=0, \end{cases}$$

Eliminating M_0 from these equations,

$$R_0=\frac{P}{(l-a)^2}\left(\frac{3l^2}{10}-al+a^2-\frac{a^4}{2l^2}+\frac{a^5}{5l^3}\right)\cdots\cdots(10)$$

and similarly

$$M_0=-\frac{P}{(l-a)^2}\left(\frac{l^3}{15}-\frac{al^2}{6}+\frac{a^3}{3}-\frac{a^4}{3l}+\frac{a^5}{10l^2}\right),\ \cdots\cdots(11)$$

whence we obtain finally

$$M_{x>a}=-\frac{P}{(l-a)^2}\left(\frac{l^3}{15}-\frac{al^2}{6}+\frac{a^3}{3}-\frac{a^4}{3l}+\frac{a^5}{10l^2}\right)$$

$$+\frac{P}{(l-a)^2}\left(\frac{3l^2}{10}-al+a^2-\frac{a^4}{2l^2}+\frac{a^5}{5l^3}\right)x\ \cdots\cdots(12)$$

$$M_{x<a}=-\frac{P}{(l-a)^2}\left(\frac{l^3}{15}-\frac{al^2}{6}+\frac{a^3}{3}-\frac{a^4}{3l}+\frac{a^5}{10l^2}\right)$$

$$+\frac{P}{(l-a)^2}\left(\frac{3l^2}{10}-al+a^2-\frac{a^4}{2l^2}+\frac{a^5}{5l^3}\right)x-\frac{P(x-a)^3}{3(l-a)^2}\ \cdots\cdots(13)$$

(General Equations of Bending Moment)

The point of the max. positive bending moment may be located by putting

$\frac{dM_x}{dx}=0$ in (13) as follows.

$$\frac{P}{(l-a)^2}\left(\frac{3l^2}{10}-al+a^2-\frac{a^4}{2l^2}+\frac{a^5}{5l^3}\right)-\frac{P}{3(l-a)^2}(3x^2-6ax+3a^2)=0$$

$$x^2-2ax-\left(\frac{3l^2}{10}-al-\frac{a^4}{2l^2}+\frac{a^5}{5l^3}\right)=0$$

from which we get

$$x=+a+\sqrt{\frac{3l^2}{10}-al+a^2-\frac{a^4}{2l^2}+\frac{a^5}{5l^3}}\ \cdots\cdots(14)$$

(Point of Max. Positive Bending Moment)

Since $\dfrac{dM_x}{dx} = S_x$ in general, from (12) and (13) we get

$$S_{x \ngtr a} = \frac{P}{(l-a)^2}\left(\frac{3l^2}{10} - al + a^2 - \frac{a^4}{2l^2} + \frac{a^5}{5l^3}\right) \quad \cdots\cdots (15)$$

$$\text{and } S_{x \nless a} = \frac{P}{(l-a)^2}\left(\frac{3l^2}{10} - al + a^2 - \frac{a^4}{2l^2} + \frac{a^5}{5l^3}\right) - \frac{P(x-a)^2}{(l-a)^2} \quad \cdots\cdots (16)$$

(General Equations of Vertical Shear)

Deduction of M_0 & R_0 by Principle of Least Work.

(Another proof)

The equations of bending moment in the beam under consideration are found from (2) & (3)

$$\begin{cases} M_{x \ngtr a} = M_0 + R_0 x \\ \& \quad M_{x \nless a} = M_0 + R_0 x - \dfrac{P(x-a)^3}{3(l-a)^2} \end{cases}$$

For the work of bending we have in general $\int_0^l \frac{M^2 dx}{2EI}$; which is expressed in the present case

$$\int_0^a [M_0 + R_0 x]^2 \frac{dx}{2EI} + \int_a^l \left[M_0 + R_0 x - \frac{P(x-a)^3}{3(l-a)^2}\right]^2 \frac{dx}{2EI}$$

$$= \int_0^a [M_0^2 + R_0^2 x^2 + 2M_0 R_0 x] \frac{dx}{2EI} + \int_a^l \left[M_0^2 + R_0^2 x^2 + \frac{P^2(x-a)^6}{9(l-a)^4}\right.$$

$$\left. - \frac{2M_0 P(x-a)^3}{3(l-a)^2} - \frac{2R_0 P x(x-a)^3}{3(l-a)^2} + 2M_0 R_0 x\right] \frac{dx}{2EI}$$

where E & I are assumed to be constant.

The values of R_0 & M_0 which make the work of bending a mini., are found by differentiating this with respect to R_0 & M_0 and putting $\dfrac{d\,(\text{work})}{dR_0} = 0$ & $\dfrac{d\,(\text{work})}{dM_0} = 0$. Thus

$$\frac{d\,(\text{work})}{dR_0}=\int_0^a [2R_0x^2+2M_0x]\frac{dx}{2EI}$$

$$+\int_a^l \left[2R_0x^2-\frac{2P}{3(l-a)^2}(x^4-3ax^3+3a^2x^2-a^3x)+2M_0x\right]\frac{dx}{2EI}=0.$$

Performing the integrations,

$$\left[\frac{2R_0x^3}{3}+\frac{2M_0x^2}{2}\right]_0^a+\left[\frac{2R_0x^3}{3}-\frac{2P}{3(l-a)^2}\left(\frac{x^5}{5}-\frac{3ax^4}{4}+\frac{3a^2x^3}{3}-\frac{a^3x^2}{2}\right)\right.$$

$$\left.+\frac{2M_0x^2}{2}\right]_a^l=0$$

$$\frac{R_0l^3}{3}+\frac{M_0l^2}{2}-\frac{P}{3(l-a)^2}\left(\frac{l^5}{5}-\frac{3al^4}{4}+a^2\,l^3-\frac{a^3l^2}{2}+\frac{a^5}{20}\right)=0 \quad \cdots\cdots\cdots\cdots(\alpha)$$

Also $$\frac{d(\text{work})}{dM_0}=\int_0^a [2M_0+2R_0x]\frac{dx}{2EI}$$

$$+\int_a^l \left[2M_0-\frac{2P}{3(l-a)^2}(x^3-3ax^2+3a^2x-a^3)+2R_0x\right]\frac{dx}{2EI}=0.$$

Performing the integrations,

$$\left[2M_0x+\frac{2R_0x^2}{2}\right]_0^a+\left[2M_0x-\frac{2P}{3(l-a)^2}\left(\frac{x^4}{4}-\frac{3ax^3}{3}+\frac{3a^2x^2}{2}-a^3x\right)+\frac{2R_0x^2}{2}\right]_a^l=0$$

$$M_0l+\frac{R_0l^2}{2}-\frac{P}{3(l-a)^2}\left(\frac{l^4}{4}-al^3+\frac{3a^2l^2}{2}-a^3l+\frac{a^4}{4}\right)=0 \quad \cdots\cdots\cdots\cdots\cdots\cdots(\beta)$$

Eliminating M_0 from (α) & (β), there is found

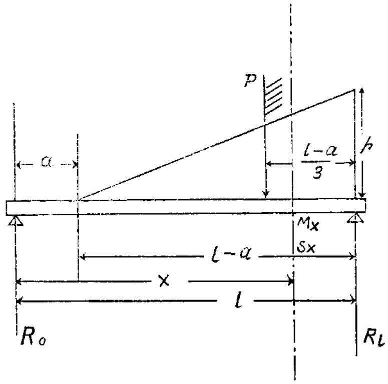
$$R_0=\frac{P}{(l-a)^2}\left(\frac{3l^2}{10}-al+a^2-\frac{a^4}{2l^2}+\frac{a^5}{5l^3}\right)$$

and similarly

$$M_0=-\frac{P}{(l-a)^2}\left(\frac{l^3}{15}-\frac{al^2}{6}+\frac{a^3}{3}-\frac{a^4}{3l}+\frac{a^5}{10l^2}\right)$$

the results being just the same as previously obtained.

B). as Beam both ends supported : —



Since the beam under consideration is supported at both ends, the equations of bending moment at any point may be represented in the following forms:

$$M_{x \ngtr a} = R_0 x \quad \cdots\cdots (17)$$

and $$M_{x \nless a} = R_0 x - \frac{P(x-a)^3}{3(l-a)^2}, \quad \cdots\cdots (18)$$

which may be obtained by putting $M_0 = 0$ in the equations (2) & (3).

Taking moment about the right end,

$$R_0 l - \frac{P(l-a)}{3} = 0,$$

whence we get

$$R_0 = \frac{P(l-a)}{3l} \quad \cdots\cdots (19)$$

and $$R_l = \frac{P(2l+a)}{3l} \quad \cdots\cdots (20)$$

Hence we obtain

$$M_{x \ngtr a} = \frac{P(l-a)x}{3l} \quad \cdots\cdots (21)$$

$$\& \ M_{x \nless a} = \frac{P(l-a)x}{3l} - \frac{P(x-a)^3}{3(l-a)^2} \quad \cdots\cdots (22)$$

(General Equations of Bending Moment)

Putting the condition $\frac{dM_x}{dx}=0$ in (22), the point of the max. bending moment may be located;

thus $$\frac{P(l-a)}{3l} - \frac{P(x-a)^2}{(l-a)^2} = 0$$

$$x^2 - 2ax - \frac{l^3 - 3al^2 - a^3}{3l} = 0$$

$$x = +a + \sqrt{\frac{(l-a)^3}{3l}} \quad \cdots\cdots (23)$$

(Point of Max. Bending Moment)

Since $\frac{dM_x}{dx} = S_x$, we get from (21) & (22)

$$S_{x \ngtr a} = \frac{P(l-a)}{3l} \quad \cdots\cdots (24)$$

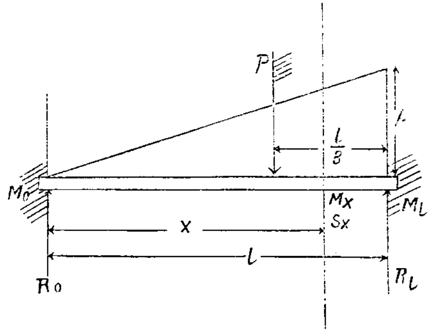
$$\text{and } S_{x \nless a} = \frac{P(l-a)}{3l} - \frac{P(x-a)^2}{(l-a)^2} \quad \cdots\cdots (25)$$

(General Equations of Vertical Shear)

— II. SPECIAL CASE —

(when $a=0$)

A). as Beam both ends fixed : —



From (3) $M_x = M_0 + R_0 x - \frac{Px^3}{3l^2}$ ....(26)

From (10) $R_0 = \frac{3P}{10}$ ....(27)

so that $R_l = \frac{7P}{10}$ ....(28)

From (11) $M_0 = -\frac{Pl}{15} = -.06667\ Pl$ ....(29)

From (13) $M_x = -\frac{Pl}{15} + \frac{3Px}{10} - \frac{Px^3}{3l^2}$ ....(30)

(General Equation of Bending Moment)

From (14) $x = \sqrt{.3}\, l = .54772\, l$ ....(31)

(Point of Max. Positive Bending Moment)

Put $x = .54772l$ in (30), then

$$M_{.547721} = .04288\,Pl \quad \cdots\cdots(32)$$

also put $x=l$, then

$$M_l = -.1\,Pl \quad \cdots\cdots(33)$$

From (16) $$S_x = \frac{3P}{10} - \frac{Px^2}{l^2} \quad \cdots\cdots(34)$$

(General Equation of Vertical Shear)

Put $x=0$ in (34), then

$$S_0 = \frac{3P}{10} \quad \cdots\cdots(35)$$

also put $x=l$, then

$$S_l = -\frac{7P}{10} \quad \cdots\cdots(36)$$

From (7) $$EI\frac{d^2y}{dx^2} = -\frac{Pl}{15} + \frac{3Px}{10} - \frac{Px^3}{3l^2} \quad \cdots\cdots(37)$$

By putting $\frac{d^2y}{dx^2}=0$, the points of inflection of the elastic curve may be located; thus

$$-\frac{Pl}{15} + \frac{3Px}{10} - \frac{Px^3}{3l^2} = 0$$

$$10x^3 - 9l^2x + 2l^3 = 0$$

from which
$$x = 0.240\,l \quad \cdots\cdots(38)$$
$$x = 0.805\,l \quad \cdots\cdots(39)$$

(Points of Inflection)

From (8) $$EI\frac{dy}{dx} = -\frac{Pl}{15}x + \frac{3Px^2}{20} - \frac{Px^4}{12l^2} \quad \cdots\cdots(40)$$

By putting $\frac{dy}{dx}=0$, the points of the max. deflection of the elastic curve may be located; thus

$$-4l^3x+9l^2x^2-5x^4=0$$

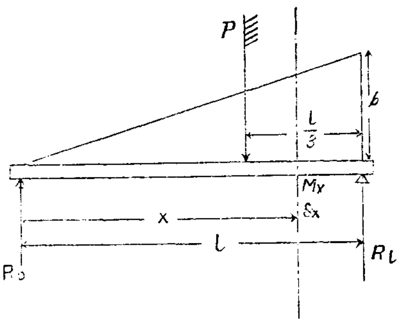
om which
$$\begin{cases} x = 0 \text{ ....} (41) \\ x = l \text{ ....} (42) \\ x = .5247\,l \text{ ....} (43) \end{cases}$$

(Points of Max. Deflection)

From (9) $EI\,y=-\frac{Plx^2}{30}+\frac{Px^3}{20}-\frac{Px^5}{60l^4}$ ....(44)

(General Equation of Elastic Curve)

B). as Beam both ends supported: —



From (19) & (20) $R_0=\frac{P}{3}$ ....(45)

& $R_l=\frac{2P}{3}$ ....(46)

From (22) $M_x=\frac{Px}{3}-\frac{Px^3}{3l^2}$ ....(47)

(General Equation of Bending Moment)

From (23) $x=\frac{l}{\sqrt{3}}=.57735\,l$ ……………………………………………………(48)

(Point of Max. Bending Moment)

Put $x=.57735\,l$ in (47), then

$M_{.57735l}=.1283\,Pl$ ……………………………………………………(49)

From (25) $S_x=\frac{P}{3}-\frac{Px^2}{l^2}$ ……………………………………………………(50)

(General Equation of Vertical Shear)

Put $x=0$ in (50), then

$S_0=\frac{P}{3}$ ……………………………………………………(51)

also put $x=l$, then

$S=-\frac{2P}{3}$ ……………………………………………………(52)
