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ON THE THEORIES OF EARTH PRESSURE.

K. Shibata, *Kogakuhakushi*.

The following is an extract of my lecture to the civil engineering students in the Imperial University of Tokyo, and may serve to show the relations of various theories on the pressure of earth.

I. Wedge Theory on Active Pressure.

1). *General solution.* (*Fig.* 1).

Let

α = inclination of the face AB of retaining wall to the horizontal,

δ = inclination of the plane of rupture AC to the horizontal,

φ = angle of repose of earth,

φ' = inclination of earth pressure P on the face AB of retaining wall to a normal on the latter,

λ = angle subtended by AC and the tangent at C to the free surface of earth,

w = weight of unit volume of earth,

k = cohesion on unit area,

l = length AC,

h = vertical height of retaining wall,

W = weight of earth ABC,

P = earth pressure on the face AB of retaining wall,

Q' = total stress on AC when $k=0$,

C = total stress on AC due to cohesion

Q = resultant of Q' and C,

taking unit length of the retaining wall and earth normal to the plane of figure.

Since W, P, C and Q' make a closed force polygon, we have

$$-P\cos(\alpha+\varphi')+C\sin\delta+Q'\cos(\delta-\varphi)=W,$$

$$P\sin(\alpha+\varphi')+C\cos\delta-Q'\sin(\delta-\varphi)=0,$$

so that

$$P = \frac{W\ sin(\delta-\varphi) - C\ cos\varphi}{sin(\alpha-\delta+\varphi+\varphi')},$$

$$Q' = \frac{W\ sin(\alpha+\varphi') - C\ cos(\alpha-\delta+\varphi')}{sin(\alpha-\delta+\varphi+\varphi')}$$

Putting the condition

$$\frac{dP}{d\delta} = 0, \ *$$

and observing that

$$C = kl,$$

$$\frac{dC}{d\delta} = -k\frac{dl}{d\delta} = -k\,l\ cot\lambda,$$

$$\frac{dW}{d\delta} = -\frac{wl^2}{2},$$

we get

$$1) \qquad W = \frac{\frac{wl^2}{2}sin(\delta-\varphi)sin(\alpha-\delta+\varphi+\varphi') - kl\frac{cos\,\varphi}{sin\,\lambda}sin(\alpha-\delta+\varphi+\varphi'-\lambda)}{sin(\alpha+\varphi')},$$

$$2) \qquad P = \frac{\frac{wl^2}{2}sin^2(\delta-\varphi) - kl\frac{cos\,\varphi}{sin\lambda}sin(\delta-\varphi+\lambda)}{sin(\alpha+\varphi')}.$$

2). *Active pressure of cohesionless earth with plane free surface.* (*Fig.* 2).

If we put

θ = angle of plane free surface of earth to the horizontal,

we have in this case

$$\lambda = \delta - \theta,$$

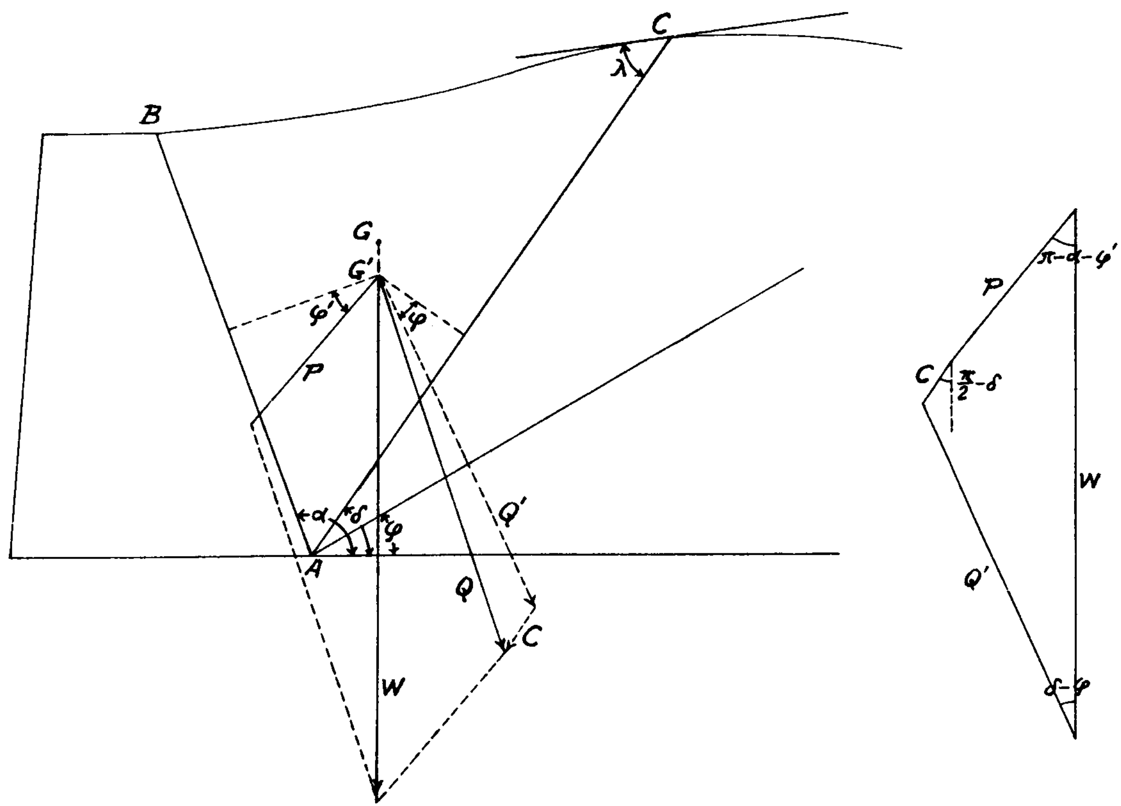
$$k = 0 \quad \text{or} \quad C = 0,$$

so that 1) and 2) become

$$3) \qquad W = \frac{wl^2}{2}\ \frac{sin(\delta-\varphi)sin(\alpha-\delta+\varphi+\varphi')}{sin(\alpha+\varphi')},$$

$$4) \qquad P = \frac{wl^2}{2}\ \frac{sin^2(\delta-\varphi)}{sin(\alpha+\varphi')} = \frac{wh^2}{2}\ \frac{sin^2(\alpha-\theta)}{sin^2\alpha\ sin(\alpha+\varphi')}\ \frac{sin^2(\delta-\varphi)}{sin^2(\delta-\theta)},$$

* Rebhann—Theorie des Erddruckes und der Futtermauern.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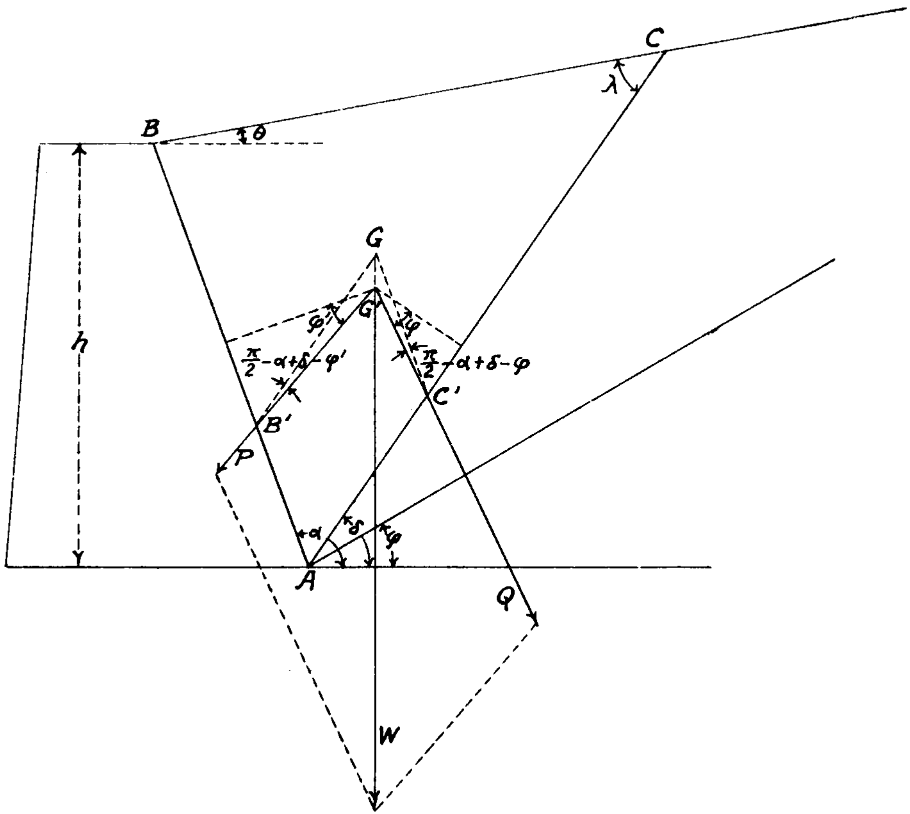

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while the condition that W is the weight of earth ABC gives

$$5)\qquad W=\frac{wl^2}{2}\ \frac{\sin(\delta-\theta)\sin(a-\delta)}{\sin(a-\theta)}.$$

Equating 3) and 5), we have

$$6)\qquad \sin(a-\theta)\sin(\delta-\varphi)\sin(a-\delta+\varphi+\varphi')=\sin(\delta-\theta)\sin(a-\delta)\sin(a+\varphi').$$

Putting

$$x=\sin(\delta-\varphi),\qquad y=\sin(\delta-\theta),$$

observing that

$$\sqrt{1-x^2}=\frac{y-x\cos(\varphi-\theta)}{\sin(\varphi-\theta)},\quad \sqrt{1-y^2}=\frac{y\cos(\varphi-\theta)-x}{\sin(\varphi-\theta)}$$

and after putting the second factor $\sin(a-\delta)$ in the right hand member of eq. 6) in the form $\sin(\overline{a-\theta}-\overline{\delta-\theta})$, eq. 6) transforms to

$$\sin(a-\theta).\ x[\sqrt{1-x^2}\sin(a+\varphi')-x\cos(a+\varphi')]$$
$$=\sin(a+\varphi').\ y[\sqrt{1-y^2}\sin(a-\theta)-y\cos(a-\theta)],$$

or

$$y^2\sin(a+\varphi')\sin(a-\varphi)-2xy\sin(a-\theta)\sin(a+\varphi')$$
$$+x^2\sin(a-\theta)\sin(a+\varphi+\varphi'-\theta)=0.$$

This is a quadratic equation of $\frac{y}{x}$, so that

$$7)\qquad \frac{y}{x}=\frac{\sin(\delta-\theta)}{\sin(\delta-\varphi)}$$

$$=\frac{\sin(a-\theta)\sin(a+\varphi')\pm\sqrt{\sin(a-\theta)\sin(a+\varphi')\sin(\varphi-\theta)\sin(\varphi+\varphi')}}{\sin(a+\varphi')\sin(a-\varphi)}.$$

To decide on the selection of double signs, put eq. 7) in the form

$$\pm\frac{\sqrt{\sin(a-\theta)\sin(a+\varphi')\sin(\varphi-\theta)\sin(\varphi+\varphi')}}{\sin(a+\varphi')\sin(a-\varphi)}=\frac{\sin(\delta-\theta)}{\sin(\delta-\varphi)}-\frac{\sin(a-\theta)}{\sin(a-\varphi)}$$

$$=\frac{\sin(a-\delta)\sin(\varphi-\theta)}{\sin(\delta-\varphi)\sin(a-\varphi)}.$$

Hence + or − sign is to be taken according as

$$8)\qquad a>\delta \qquad\text{or}\qquad a<\delta.$$

In the present case $a\geqq\delta$, so that

$$\frac{\sin(\delta-\theta)}{\sin(\delta-\varphi)}=\frac{\sin(a-\theta)\sin(a+\varphi')+\sqrt{\sin(a-\theta)\sin(a+\varphi')\sin(\varphi-\theta)\sin(\varphi+\varphi')}}{\sin(a+\varphi')\sin(a-\varphi)},$$

whence

$$\frac{\sin(\delta-\theta)}{\sin(a-\theta)\sin(\delta-\varphi)}=\frac{\sqrt{\sin(a-\theta)\sin(a+\varphi')}+\sqrt{\sin(\varphi-\theta)\sin(\varphi+\varphi')}}{\sin(a-\varphi)\sqrt{\sin(a-\theta)\sin(a+\varphi')}},$$

Substituting last expression in eq. 4), we have

$$9)\quad P=\frac{wh^2}{2}\cdot\frac{\sin(a-\theta)}{\sin^2 a}\cdot\frac{\sin^2(a-\varphi)}{[\sqrt{\sin(a-\theta)\sin(a+\varphi')}+\sqrt{\sin(\varphi-\theta)\sin(\varphi+\varphi')}]^2},$$

while eq. 7.) gives, after some simple reductions,

$$10)\quad tg\delta=\frac{\sin\varphi\sqrt{\sin(a-\theta)\sin(\varphi+\varphi')}+\sin a\sqrt{\sin(\varphi-\theta)\sin(a+\varphi')}}{\cos\varphi\sqrt{\sin(a-\theta)\sin(\varphi+\varphi')}+\cos a\sqrt{\sin(\varphi-\theta)\sin(a+\varphi')}}.$$

Eq. 10) shows that, when $a=\delta$, we have $\delta=\varphi$, as it should be.

II. Weyrauch's Formulae on Active Pressure.

Confining our attention to the case of the active pressure of cohesionless earth with plane free surface, if we assume that the equilibrium condition of earth is not affected by the introduction of a retaining wall, we must have

$$3AB'=AB,\qquad 3AC'=AC.$$

Thus it follows that, if G is the centre of gravity of the triangle ABC,

$$\frac{AB}{AC}=\frac{\sin(\delta-\theta)}{\sin(a-\theta)}$$

$$=\frac{AB'}{AC'}=\frac{GC'}{GB'}=\frac{GG'\sin(\delta-\varphi)}{\cos(a-\delta+\varphi)}\cdot\frac{\cos(a-\delta+\varphi')}{GG'\sin(a+\varphi')},$$

so that

$$11)\quad \sin(a-\theta)\sin(\delta-\varphi)\cos(a-\delta+\varphi')=\sin(\delta-\theta)\cos(a-\delta+\varphi)\sin(a+\varphi').$$

Equations 6) and 11) serve to determine δ and φ'. Thus put these equations in the form

$$-\sin(\delta-\varphi)\cot(a+\varphi')+\cos(\delta-\varphi)=\frac{\sin(\delta-\theta)\sin(a-\delta)}{\sin(a-\theta)\sin(\delta-\varphi)},$$

$$\cos\delta\cot(a+\varphi')+\sin\delta=\frac{\sin(\delta-\theta)\cos(a-\delta+\varphi)}{\sin(a-\theta)\sin(\delta-\varphi)}.$$

Eliminating $\cot(a+\varphi')$ from these equations we get

$$\cos\varphi=\frac{\sin(\delta-\theta)[\sin(a-\delta)\cos\delta+\cos(a-\delta+\varphi)\sin(\delta-\varphi)]}{\sin(a-\theta)\sin(\delta-\varphi)},$$

or

$$\sin(\delta-\theta)\sin(a-\delta)\cos\delta=\sin(\delta-\varphi)\left[\cos\varphi\sin(a-\theta)-\sin(\delta-\theta)\cos(a-\delta+\varphi)\right]$$
$$=\sin(\delta-\varphi)\sin(a-\delta)\cos(\delta-\varphi-\theta).$$

Thus we have

$$12)\qquad \delta=a$$

as one solution. Another solution is

$$\sin(\delta-\theta)\cos\delta=\sin(\delta-\varphi)\cos(\delta-\varphi-\theta),$$

which gives, after some simple reductions,

$$13)\begin{cases}\sin\varphi\cos(2\delta-\varphi-\theta)=\sin\theta\\ \text{or } 2\delta=\frac{\pi}{2}+\varphi+\theta-\psi,\quad \text{where } \sin\psi=\frac{\sin\theta}{\sin\varphi},\quad \cos\psi=\frac{\sqrt{\cos^2\theta-\cos^2\varphi}}{\sin\varphi}.\end{cases}$$

Nextly, to find φ', eqs. 6) and 11) give

$$\cos(a-\delta+\varphi')\sin(a-\delta)=\cos(a-\delta+\varphi)\sin(a-\delta+\varphi+\varphi').$$

Hence corresponding to the value of δ given by 12), we have

$$14)\qquad \varphi'=-\varphi.$$

To find φ' corresponding to eq. 13) put above equation in the form

$$\sin\varphi'\left[\sin^2(a-\delta)+\cos^2(a-\delta+\varphi)\right]$$
$$=\cos\varphi'\left[\sin(a-\delta)\cos(a-\delta)-\sin(a-\delta+\varphi)\cos(a-\delta+\varphi)\right],$$

or

$$tg\,\varphi'=\frac{-\sin\varphi\cos(2a-2\delta+\varphi)}{1-\sin\varphi\sin(2a-2\delta+\varphi)}.$$

Substituting the value of 2δ in eq. 13), we obtain

$$15)\begin{cases}tg\,\varphi'=\frac{-\sin\varphi\sin(2a-\theta+\psi)}{1+\sin\varphi\cos(2a-\theta+\psi)}\\ \quad=\frac{-\cos(2a-\theta)\sin\theta-\sin(2a-\theta)\sqrt{\cos^2\theta-\cos^2\varphi}}{1-\sin(2a-\theta)\sin\theta+\cos(2a-\theta)\sqrt{\cos^2\theta-\cos^2\varphi}}.\end{cases}$$

The expression for P is given by eq. 9), while the values of a less than δ must be considered as admissible in this case.

Hence we have

$$16)\begin{cases}P=\frac{wh^2}{2}\cdot\frac{\sin(a-\theta)}{\sin^2 a}\\ \qquad\times\frac{\sin^2(a-\varphi)}{\left[\sqrt{\sin(a-\theta)\sin(a+\varphi')}+\sqrt{\sin(\varphi-\theta)\sin(\varphi+\varphi')}\right]^2}\qquad a\geqq\delta\end{cases}$$

$$P=\frac{wh^2}{2}\cdot\frac{\sin(a-\theta)}{\sin^2 a}$$
$$\times\frac{\sin^2(a-\varphi)}{[\sqrt{\sin(a-\theta)\sin(a+\varphi')}-\sqrt{\sin(\varphi-\theta)\sin(\varphi+\varphi')}]^2}\qquad a\leqq\delta.$$

To put these equations in a form not containing φ', we have, by eq. 15),

$$\sin(a+\varphi')=\frac{\sin a-\sin\varphi\sin(a-\theta+\psi)}{\sqrt{D}}$$

$$\sin(\varphi+\varphi')=\frac{\sin\varphi-\sin\varphi\sin(2a-\theta-\varphi+\psi)}{\sqrt{D}},$$

where

17) $$D\equiv 1+\sin^2\varphi+2\sin\varphi\cos(2a-\theta+\psi)$$
$$=1+\sin^2\varphi-2\sin(2a-\theta)\sin\theta+2\cos(2a-\theta)\sqrt{\cos^2\theta-\cos^2\varphi}.$$

Thus

$$\sin(a-\theta)\sin(a+\varphi')$$
$$=\frac{1}{\sqrt{D}}\sin(a-\theta)[\sin a-\cos(a-\theta)\sin\theta-\sin(a-\theta)\sqrt{\cos^2\theta-\cos^2\varphi}]$$
$$=\frac{1}{\sqrt{D}}\sin^2(a-\theta)(\cos\theta-\sqrt{\cos^2\theta-\cos^2\varphi})$$
$$=\frac{\sin^2(a-\theta)\cos^2\varphi}{\sqrt{D}(\cos\theta+\sqrt{\cos^2\theta-\cos^2\varphi})}.$$

$$\sin(\varphi-\theta)\sin(\varphi+\varphi')$$
$$=\frac{1}{\sqrt{D}}\sin(\varphi-\theta)[\sin\varphi-\cos(2a-\theta-\varphi)\sin\theta-\sin(2a-\theta-\varphi)\sqrt{\cos^2\theta-\cos^2\varphi}.$$

The expression in square brackets on the right hand member of last equation becomes, when it is multiplied by $\cos\theta+\sqrt{\cos^2\theta-\cos^2\varphi}$,

$$\sin\varphi\cos\theta-\cos(2a-\theta-\varphi)\sin\theta\cos\theta-\sin(2a-\theta-\varphi)(\cos^2\theta-\cos^2\varphi)$$
$$+[\sin\varphi-\sin(2a-\theta-\varphi)\cos\theta-\cos(2a-\theta-\varphi)\sin\theta]\sqrt{\cos^2\theta-\cos^2\varphi}$$
$$=\sin\varphi\cos\theta-\sin(2a-\varphi)\cos\theta+\sin(2a-\varphi)\cos\theta\cos^2\varphi-\cos(2a-\varphi)\sin\theta\cos^2\varphi$$
$$+[\sin\varphi-\sin(2a-\varphi)]\sqrt{\cos^2\theta-\cos^2\varphi}$$
$$=\sin\varphi\cos\theta[1-\sin(2a-\varphi)\sin\varphi]-\cos\varphi\sin\theta\cos(2a-\varphi)\cos\varphi$$

$$-2\sin(\alpha-\varphi)\cos\alpha\sqrt{\cos^2\theta-\cos^2\varphi}$$

$$=\sin^2(\alpha-\varphi)\sin(\varphi+\theta)+\cos^2\alpha\sin(\varphi-\theta)-2\sin(\alpha-\varphi)\cos\alpha\sqrt{\cos^2\theta-\cos^2\varphi}$$

$$=[\sin(\alpha-\varphi)\sqrt{\sin(\varphi+\theta)}-\cos\alpha\sqrt{\sin(\varphi-\theta)}]^2,$$

so that

$$\sin(\varphi-\theta)\sin(\varphi+\varphi')=\frac{[\sin(\alpha-\varphi)\sqrt{\cos^2\theta-\cos^2\varphi}-\cos\alpha\sin(\varphi-\theta)]^2}{\sqrt{D}(\cos\theta+\sqrt{\cos^2\theta-\cos^2\varphi})}.$$

We observe that the expression within the square brackets in the numerator of the right hand member of last equation is positive or negative according as

$$\sin(\alpha-\varphi)\sqrt{\sin(\varphi+\theta)}\gtrless\cos\alpha\sqrt{\sin(\varphi-\theta)},$$

or

$$[1-\cos(2\alpha-2\varphi)]\sin(\varphi+\theta)\gtrless(1+\cos 2\alpha)\sin(\varphi-\theta),$$

which condition becomes, after some easy reductions,

$$\cos(2\alpha-\varphi-\theta)\lessgtr\frac{\sin\theta}{\sin\varphi},$$

or, comparing with eq. 13),

$$\alpha\gtrless\delta.$$

With this condition in view, we obtain

$$\sqrt{\sin(\alpha-\theta)\sin(\alpha+\varphi')}\pm\sqrt{\sin(\varphi-\theta)\sin(\varphi+\varphi')}$$

$$=\frac{\sin(\alpha-\theta)\cos\varphi-\cos\alpha\sin(\varphi-\theta)+\sin(\alpha-\varphi)\sqrt{\cos^2\theta-\cos^2\varphi}}{\sqrt[4]{D}\sqrt{\cos\theta+\sqrt{\cos^2\theta-\cos^2\varphi}}}$$

$$=\frac{\sin(\alpha-\varphi)}{\sqrt[4]{D}}\sqrt{\cos\theta+\sqrt{\cos^2\theta-\cos^2\varphi}}.$$

Substituting this value in eq. 16), and putting in the expression 17), we arrive at

18) $$P=\frac{wl^2}{2}\cdot\frac{\sin(\alpha-\theta)}{\sin^2\alpha}$$

$$\times\frac{\sqrt{1+\sin^2\varphi-2\sin(2\alpha-\theta)\sin\theta+2\cos(2\alpha-\theta)\sqrt{\cos^2\theta-\cos^2\varphi}}}{\cos\theta+\sqrt{\cos^2\theta-\cos^2\varphi}}.$$

III. Rankine's Formulae on Active Pressure Generalized.

Substituting the values of conjugate stresses, (Fig. 3)

$$19)\begin{cases} T_1 = w\,y\cos\theta \\ T_2 = T_1\dfrac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\varphi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\varphi}} \end{cases}$$

to the formulae giving the principal stresses a and b and principal axes, we have

$$a = T_1\frac{1 - \sin\varphi}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\varphi}}$$

$$b = T_1\frac{1 + \sin\varphi}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\varphi}}$$

$$\cos 2\beta = \cos(2\beta_2 + 2a - \pi) = \cos(\psi - \theta + 2a - \pi)$$

$$= -\cos(2a - \theta + \psi) = \frac{\sin(2a-\theta)\sin\theta - \cos(2a-\theta)\sqrt{\cos^2\theta - \cos^2\varphi}}{\sin\varphi},$$

so that the stress p on the plane making the angle a with the horizontal is

$$p = \sqrt{a^2\cos^2\beta + b^2\sin^2\beta} = \sqrt{\frac{b^2 + a^2}{2} - \frac{b^2 - a^2}{2}\cos 2\beta}$$

$$= T_1\frac{\sqrt{1 + \sin^2\varphi - 2\sin(2a-\theta)\sin\theta + 2\cos(2a-\theta)\sqrt{\cos^2\theta - \cos^2\varphi}}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\varphi}}$$

Hence observing that

$$y = \eta\frac{\sin(a-\theta)}{\sin a\cos\theta}$$

$$P = \int_0^h p\frac{d\eta}{\sin a},$$

we obtain

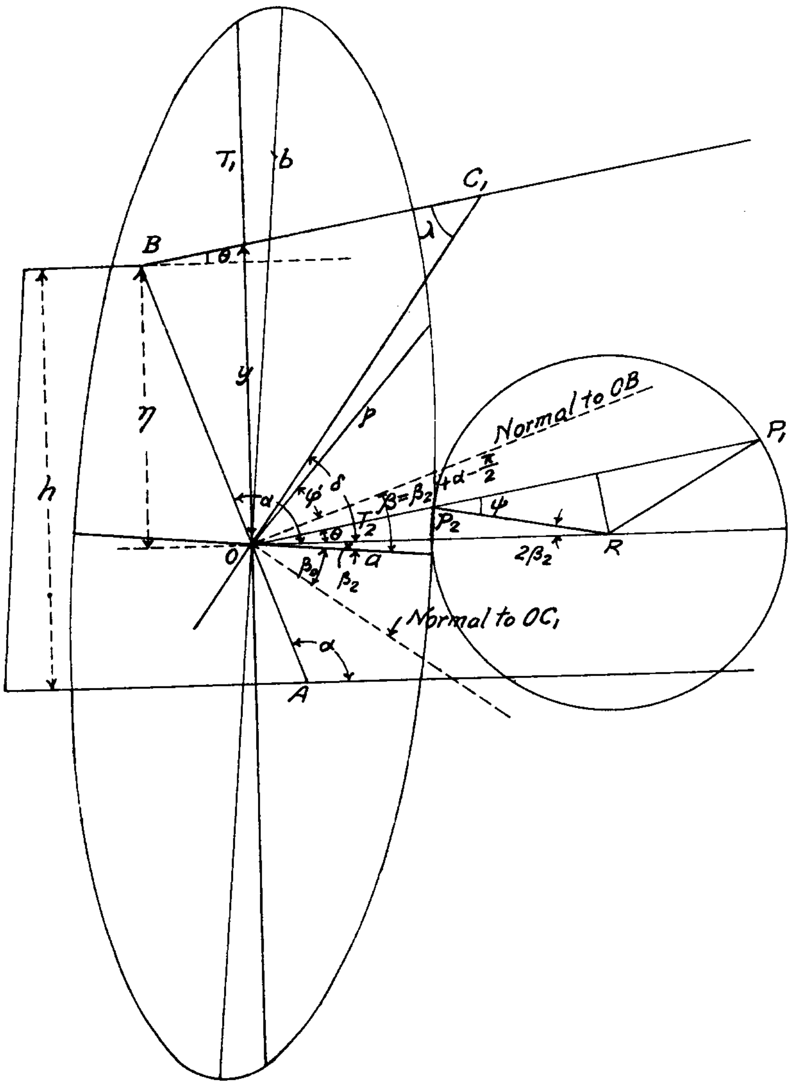
$$20)\quad P = \frac{wh^2}{2}\cdot\frac{\sin(a-\theta)}{\sin^2 a}$$

$$\times\frac{\sqrt{1 + \sin^2\varphi - 2\sin(2a-\theta)\sin\theta + 2\cos(2a-\theta)\sqrt{\cos^2\theta - \cos^2\varphi}}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\varphi}}$$

which is the same as eq. 18).

To find φ', we have, by a well-known theorem on stress analysis,

$$tg\,\varphi' = \frac{(b-a)\sin\beta\cos\beta}{a\cos^2\beta + b\sin^2\beta} = \frac{(b-a)\sin 2\beta}{b + a - (b-a)\cos 2\beta},$$



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Normal to OB
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T2
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2β2
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Normal to OC1
α
A

which gives

$$21)\left\{\begin{aligned} tg\,\varphi' &= \frac{-\sin\varphi\,\sin(2a-\theta+\psi)}{1+\sin\varphi\,\cos(2a-\theta+\psi)} \\ &= \frac{-\cos(2a-\theta)\sin\theta-\sin(2a-\theta)\sqrt{\cos^2\theta-\cos^2\varphi}}{1-\sin(2a-\theta)\sin\theta+\cos(2a-\theta)\sqrt{\cos^2\theta-\cos^2\varphi}}.\end{aligned}\right.$$

This is the same equation as 15).

Lastly, to find δ, we have to determine the value of δ for a plane, the stress on which makes the angle φ with its normal. This condition gives, for the angle β_0 made by the normal on the plane with a-axis,

$$\beta_0=\frac{\pi}{4}-\frac{\varphi}{2}.$$

Hence

$$\begin{aligned} 2\delta &= 2\left(\frac{\pi}{2}-\beta_0-\beta_2\right) \\ &= 2\left(\frac{\pi}{2}-\frac{\pi}{4}+\frac{\varphi}{2}-\beta_2\right) \\ &= \varphi+\frac{\pi}{2}-2\beta_2 \\ &= \varphi+\frac{\pi}{2}-\psi+\theta, \end{aligned}$$

which is the same as eq. 13).

IV. Conclusion on Active Pressure.

It is long since it has been pointed out that the wedge theory is *never* founded on the "Principle of earth prism of greatest pressure".* We have shewn above that *Weyrauch's formulae are identical with those of Rankine,* and we can easily prove that *the formulae given in Howe's "Retaining Walls for Earth" are also identical with them.* The methods followed out in deducing the formulae of these three authorities are not identical, but *they are founded on exactly the same basis.* The *only difference* of wedge theory and Weyrauch-Rankine's theory is that the condition of equilibrium of earth is assumed in the former to be modified by the introduction of a retaining wall, while in the latter no such

* Rebhann-Theorie des Erddruckes und der Futtermauern, p. 44.

is considered to take place. With this exception these two theories are one and the same, as the present deductions above given will amply elucidate the fact.

It is too well known to add that some diversities of opinion exist in wedge theory with regard to the value to be given to φ'. This is a question which needs not to be touched here.

V. Wedge Theory on Passive Pressure.

1). *General Solution.* (*Fig.* 4).

Adopting the same notations as in the case of active pressure, we have

$$-P\cos(\alpha-\varphi')-C\sin\delta+Q'\cos(\delta+\varphi)=W,$$

$$P\sin(\alpha-\varphi')-C\cos\delta-Q'\sin(\delta+\varphi)=0.$$

Comparing these wrih corresponding equations for active pressure, we see that the former are obtained from the latter simply by changing the signs of φ,φ' and C. Hence corresponding to eqs. 1) and 2) we have

$$(1)\quad W=\frac{\frac{wl^2}{2}\sin(\delta+\varphi)\sin(\alpha-\delta-\varphi-\varphi')+kl\frac{\cos\varphi}{\sin\lambda}\sin(\alpha-\delta-\varphi-\varphi'-\lambda)}{\sin(\alpha-\varphi')},$$

$$(2)\quad P=\frac{\frac{wl^2}{2}\sin^2(\delta+\varphi)+kl\frac{\cos\varphi}{\sin\lambda}\sin(\delta+\varphi+\lambda)}{\sin(\alpha-\varphi')}.$$

2). *Passive pressure of cohesionless earth with plane free surface.* (*Fig.* 5).
Here corresponding to eqs. 3) to 7), we obtain

$$(3)\quad W=\frac{wl^2}{2}\,\frac{\sin(\delta+\varphi)\sin(\alpha-\delta-\varphi-\varphi')}{\sin(\alpha-\varphi')},$$

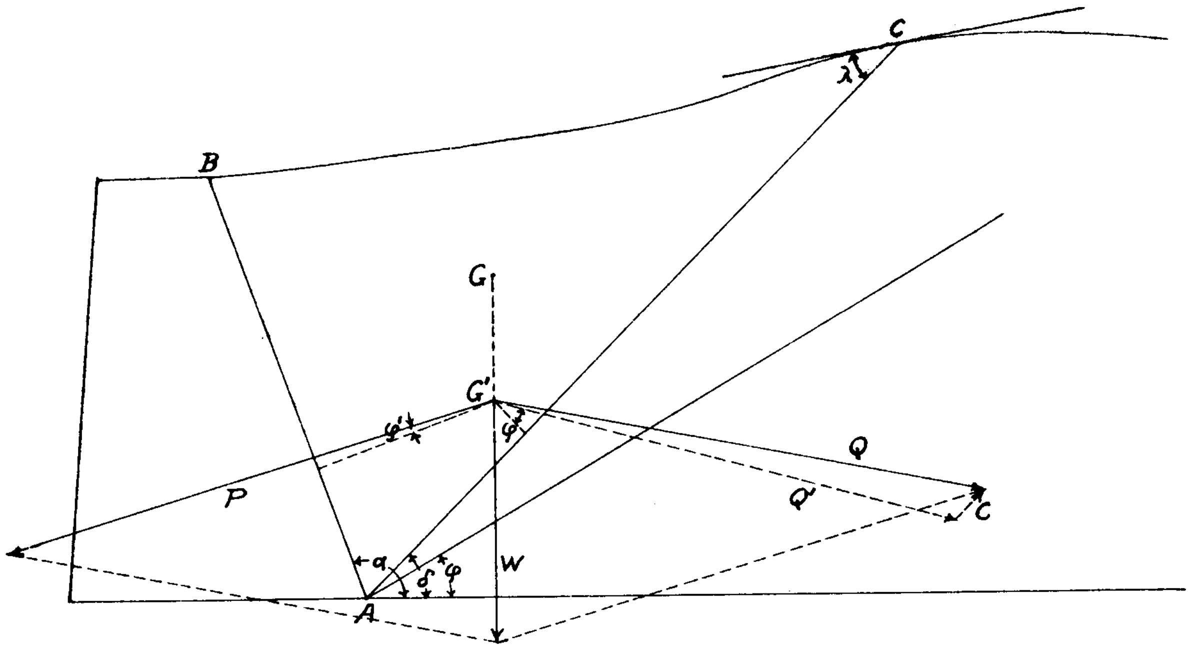
$$(4)\quad P=\frac{wl^2}{2}\,\frac{\sin^2(\delta+\varphi)}{\sin(\alpha-\varphi')}=\frac{wh^2}{2}\,\frac{\sin^2(\alpha-\theta)}{\sin^2\alpha\sin(\alpha-\varphi')}\,\frac{\sin^2(\delta+\varphi)}{\sin^2(\delta-\theta)},$$

$$(5)\quad W=\frac{wl^2}{2}\,\frac{\sin(\delta-\theta)\sin(\alpha-\delta)}{\sin(\alpha-\theta)},$$

$$(6)\quad \sin(\alpha-\theta)\sin(\delta+\varphi)\sin(\alpha-\delta-\varphi-\varphi')=\sin(\delta-\theta)\sin(\alpha-\delta)\sin(\alpha-\varphi'),$$

$$(7)\quad \frac{\sin(\delta-\theta)}{\sin(\delta+\varphi)}$$

$$=\frac{\sin(\alpha-\theta)\sin(\alpha-\varphi')\pm\sqrt{\sin(\alpha-\theta)\sin(\alpha-\varphi')\sin(\varphi+\theta)\sin(\varphi+\varphi')}}{\sin(\alpha-\varphi')\sin(\alpha+\varphi)}.$$



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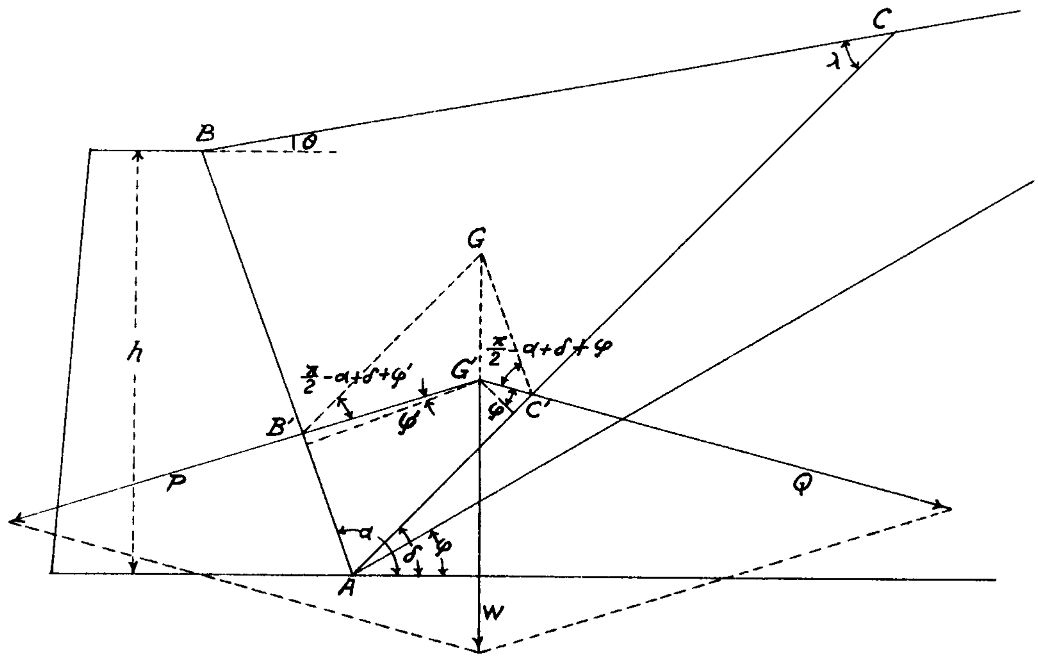

C
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$\frac{\pi}{2}-\alpha+\delta+\varphi$
$\frac{\pi}{2}-\alpha+\delta+\varphi'$
G'
B'
φ'
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C'
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Q
α
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φ
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W

To decide on the selection of double signs in eq. (7), put it in the form

$$\pm\frac{\sqrt{sin(\alpha-\theta)sin(\alpha-\varphi')sin(\varphi+\theta)sin(\varphi+\varphi')}}{sin(\alpha-\varphi')sin(\alpha+\varphi)}=\frac{sin(\delta-\theta)}{sin(\delta+\varphi)}-\frac{sin(\alpha-\theta)}{sin(\alpha+\varphi)}$$

$$=-\frac{sin(\alpha-\delta)sin(\varphi+\theta)}{sin(\delta+\varphi)sin(\alpha+\varphi)}.$$

Hence + or − sign is to be taken according as

(8) $\alpha<\delta$ or $\alpha>\delta$.

In the present case $\alpha\geqq\delta$, so that

$$\frac{sin(\delta-\theta)}{sin(\delta+\varphi)}$$

$$=\frac{sin(\alpha-\theta)sin(\alpha-\varphi')-\sqrt{sin(\alpha-\theta)sin(\alpha-\varphi')sin(\varphi+\theta)sin(\varphi+\varphi')}}{sin(\alpha-\varphi')sin(\alpha+\varphi)},$$

whence

$$\frac{sin(\delta-\theta)}{sin(\alpha-\theta)sin(\delta+\varphi)}=\frac{\sqrt{sin(\alpha-\theta)sin(\alpha-\varphi')}-\sqrt{sin(\varphi+\theta)sin(\varphi+\varphi')}}{sin(\alpha+\varphi)\sqrt{sin(\alpha-\theta)sin(\alpha-\varphi')}}.$$

Substituting this expression in eq. (4), we have

(9) $$P=\frac{wh^2}{2}\cdot\frac{sin(\alpha-\theta)}{sin^2\alpha}\cdot\frac{sin^2(\alpha+\varphi)}{[\sqrt{sin(\alpha-\theta)sin(\alpha-\varphi')}-\sqrt{sin(\varphi+\theta)sin(\varphi+\varphi')}]^2},$$

while eq. (7) gives, after some simple reductions,

(10) $$tg\ \delta=\frac{-sin\,\varphi\sqrt{sin(\alpha-\theta)sin(\varphi+\varphi')}+sin\,\alpha\sqrt{sin(\varphi+\theta)sin(\alpha-\varphi')}}{cos\,\varphi\sqrt{sin(\alpha-\theta)sin(\varphi+\varphi')}+cos\,\alpha\sqrt{sin(\varphi+\theta)sin(\alpha-\varphi')}}.$$

Eq. (10) shows that, if $\varphi'=\theta$, then

$$\alpha-\delta=\delta+\varphi \quad \text{or} \quad 2\delta=\alpha-\varphi,$$

so that in case $\alpha=3\varphi$, we shall have $\delta=\varphi$.

VI. Weyrauch's Formulae on Passive Pressure.

Corresponding to eq. 11), we have in this case

(11) $$sin(\alpha-\theta)sin(\delta+\varphi)cos(\alpha-\delta-\varphi')=sin(\delta-\theta)cos(\alpha-\delta-\varphi)sin(\alpha-\varphi').$$

This equation combined with eq. (6) gives, in a similar manner as in the case of active pressure,

$$sin(\delta-\theta)sin(\alpha-\delta)cos\,\delta=sin(\delta+\varphi)sin(\alpha-\delta)cos(\delta+\varphi-\theta),$$

Thus we have

$$(12) \qquad \delta = a$$

as one solution. Another solution is

$$\sin(\delta-\theta)\cos\delta = \sin(\delta+\varphi)\cos(\delta+\varphi-\theta),$$

which gives, after some simple reductions,

$$(13)\begin{cases} \sin\varphi\cos(2\delta+\varphi-\theta) = -\sin\theta \\ \text{or } 2\delta = \dfrac{\pi}{2} - \varphi + \theta + \psi, \text{ where } \sin\psi = \dfrac{\sin\theta}{\sin\varphi}, \quad \cos\psi = \dfrac{\sqrt{\cos^2\theta - \cos^2\varphi}}{\sin\varphi}. \end{cases}$$

Nextly, to find φ', eqs. (6) and (11) give

$$\cos(a-\delta-\varphi')\sin(a-\delta) = \cos(a-\delta-\varphi)\sin(a-\delta-\varphi-\varphi').$$

Hence corresponding to the value of δ given by (12), we have

$$(14) \qquad \varphi' = -\varphi.$$

To find φ' corresponding to eq. (13), above equation will give, in a similar way as in the case of active pressure,

$$(15)\begin{cases} tg\ \varphi' = \dfrac{-\sin\varphi\sin(2a-\theta-\psi)}{1-\sin\varphi\cos(2a-\theta-\psi)} \\ \qquad = \dfrac{\cos(2a-\theta)\sin\theta - \sin(2a-\theta)\sqrt{\cos^2\theta-\cos^2\varphi}}{1-\sin(2a-\theta)\sin\theta-\cos(2a-\theta)\sqrt{\cos^2\theta-\cos\varphi}} \end{cases}$$

The expression for P is given by eq. (9), while the values of a less than δ must be considered as admissible in this case. Hence we have

$$(16)\begin{cases} P = \dfrac{wh^2}{2}\cdot\dfrac{\sin(a-\theta)}{\sin^2 a} \\ \qquad \times \dfrac{\sin^2(a+\varphi)}{[\sqrt{\sin(a-\theta)\sin(a-\varphi')} - \sqrt{\sin(\varphi+\theta)\sin(\varphi+\varphi')}]^2} \qquad a \geqq \delta \\ P = \dfrac{wh^2}{2}\cdot\dfrac{\sin(a-\theta)}{\sin^2 a} \\ \qquad \times \dfrac{\sin^2(a+\varphi)}{[\sqrt{\sin(a-\theta)\sin(a-\varphi')} + \sqrt{\sin(\varphi+\theta)\sin(\varphi+\varphi')}]^2} \qquad a \leqq \delta \end{cases}$$

To transform these equations in a form not containing φ', we shall have, in a similar manner as in the case of active pressure,

$$(17)\quad D \equiv 1+\sin^2\varphi-2\sin\varphi\cos(2\alpha-\theta-\psi)$$

$$=1+\sin^2\varphi-2\sin(2\alpha-\theta)\sin\theta-2\cos(2\alpha-\theta)\sqrt{\cos^2\theta-\cos^2\varphi},$$

$$\sin(\alpha-\theta)\sin(\alpha-\varphi')=\frac{\sin^2(\alpha-\theta)\cos^2\varphi}{\sqrt{D}\,(\cos\theta-\sqrt{\cos^2\theta-\cos^2\varphi})}$$

$$\sin(\varphi+\theta)\sin(\varphi+\varphi')=\frac{[-\sin(\alpha+\varphi)\sqrt{\cos^2\theta-\cos^2\varphi}+\cos\alpha\sin(\varphi+\theta)]^2}{\sqrt{D}\,(\cos\theta-\sqrt{\cos^2\theta-\cos^2\varphi})}$$

Observing that

$$-\sin(\alpha+\varphi)\sqrt{\cos^2\theta-\cos^2\varphi}+\cos\alpha\,\sin(\varphi+\theta)\lesseqgtr 0$$

according as

$$\alpha \gtrless \delta,$$

which can be proved as in the case of active pressure, we have

$$\sqrt{\sin(\alpha-\theta)\sin(\alpha-\varphi')}\mp\sqrt{\sin(\varphi+\theta)\sin(\varphi+\varphi')}$$

$$=\frac{\sin(\alpha-\theta)\cos\varphi-\sin(\alpha+\varphi)\sqrt{\cos^2\theta-\cos^2\varphi}+\cos\alpha\sin(\varphi+\theta)}{\sqrt[4]{D}\sqrt{\cos\theta-\sqrt{\cos^2\theta-\cos^2\varphi}}}$$

$$=\frac{\sin(\alpha+\varphi)}{\sqrt[4]{D}}\sqrt{\cos\theta-\sqrt{\cos^2\theta-\cos^2\varphi}}$$

Substituting this value in eq. (16), and putting in the expression (17), we obtain

$$(18)\quad P=\frac{wh^2}{2}\cdot\frac{\sin(\alpha-\theta)}{\sin^2\alpha}$$

$$\times\frac{\sqrt{1+\sin^2\varphi-2\sin(2\alpha-\theta)\sin\theta-2\cos(2\alpha-\theta)\sqrt{\cos^2\theta-\cos^2\varphi}}}{\cos\theta-\sqrt{\cos^2\theta-\cos^2\varphi}}$$

VII. Rankine's Formulae on Passive Pressure Generalized.

Here corresponding to eq. 19), we have. (Fig. 6)

$$(19)\quad \begin{cases} T_1=wy\cos\theta \\ T_2=T_1\dfrac{\cos\theta+\sqrt{\cos^2\theta-\cos^2\varphi}}{\cos\theta-\sqrt{\cos^2\theta-\cos^2\varphi}} \end{cases}$$

so that the prinicipal stresses a and b and principal axes are given by

$$a=T_1\frac{1-\sin\varphi}{\cos\theta-\sqrt{\cos^2\theta-\cos^2\varphi}}$$

$$b = T_1 \frac{1+\sin\varphi}{\cos\theta - \sqrt{\cos^2\theta - \cos^2\varphi}}$$

$$\cos 2\beta = \cos(2\beta_2 + 2a - \pi) = \cos(2a - \theta - \psi)$$

$$= \frac{\sin(2a-\theta)\sin\theta + \cos(2a-\theta)\sqrt{\cos^2\theta - \cos^2\varphi}}{\sin\varphi}$$

Hence the stress p on the plane making the angle a with the horizontal is

$$p = \sqrt{a^2\cos^2\beta + b^2\sin^2\beta}$$

$$= T_1 \frac{\sqrt{1+\sin^2\varphi - 2\sin(2a-\theta)\sin\theta - 2\cos(2a-\theta)\sqrt{\cos^2\theta - \cos^2\varphi}}}{\cos\theta - \sqrt{\cos^2\theta - \cos^2\varphi}}$$

whence in a similar manner as in the case of active pressure we have

$$(20) \quad P = \frac{wh^2}{2} \cdot \frac{\sin(a-\theta)}{\sin^2 a}$$

$$\times \frac{\sqrt{1+\sin^2\varphi - 2\sin(2a-\theta)\sin\theta - 2\cos(2a-\theta)\sqrt{\cos^2\theta - \cos^2\varphi}}}{\cos\theta - \sqrt{\cos^2\theta - \cos^2\varphi}},$$

which is the same as eq. (18).

To find φ', we have, in a similar way as in the case of active pressure,

$$(21) \quad \begin{cases} tg\,\varphi' = \dfrac{-\sin\varphi\sin(2a-\theta-\psi)}{1-\sin\varphi\cos(2a-\theta-\psi)} \\[2ex] \quad = \dfrac{\cos(2a-\theta)\sin\theta - \sin(2a-\theta)\sqrt{\cos^2\theta - \cos^2\varphi}}{1-\sin(2a-\theta)\sin\theta - \cos(2a-\theta)\sqrt{\cos^2\theta - \cos^2\varphi}}, \end{cases}$$

which is the same equation as (15).

Lastly, to find δ, we have

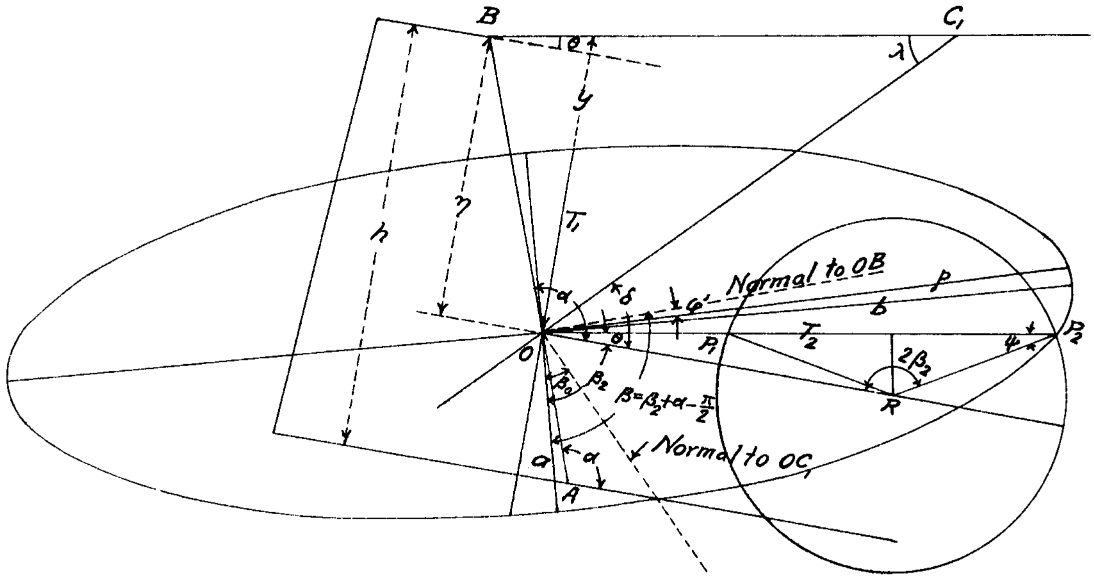
$$2\delta = 2\left(\frac{\pi}{2} + \beta_0 - \beta_2\right)$$

$$= 2\left(\frac{\pi}{2} + \frac{\pi}{4} - \frac{\varphi}{2} - \beta_2\right)$$

$$= \frac{3\pi}{2} - \varphi - 2\beta_2$$

$$= \frac{\pi}{2} - \varphi + \theta + \psi$$

which is the same as eq. (13).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