A method of Locating the True Equilibrium Polygon on an Hingeless Arch.

In the method presented below, an attempt has been made to avoid the usual practice to divide the arch axis, so that $\frac{\Delta s}{I}$ is constant,* which involves no little labor. Moreover, the method of establishing the closing line on the trial polygon may be probably new.

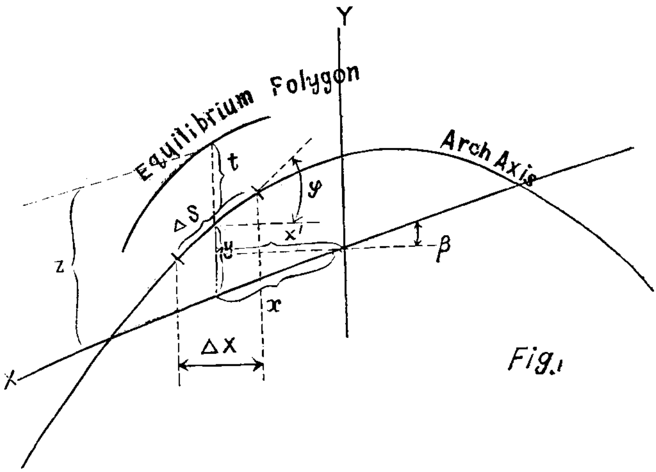

Fig. 1

Fig. 1 shows an unsymmetrical arch axis and a portion of equilibrium polygon.

Neglecting the effect of axial thrust, the fundamental equations for a hingeless arch, refered to any axis, may be expressed in general form:

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* Cain-Theory of Solid & Braced Elastic Arches. p. 19. Ketchum-Design of Highway Bridges. p. 343. Buel & Hill-Reinforced Concrete. p. 133. Turneaure & Maurer-Principle of reinforced Concrete Construction. p 272. Reuterdahl-Theory and Design of Reinforced Concrete Construction. p. 11 etc.

$$\left.\begin{aligned}\Sigma\frac{M\triangle S}{EI}&=0\\ \Sigma\frac{Mx\triangle S}{EI}&=0\\ \Sigma\frac{My\triangle S}{EI}&=0\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

Take Y–axis vertical, X–axis inclind β to the horizontal and the origin at any point for the present.

Let I_c be any constant moment of inertia, preferably that at the crown, and φ the inclination of arch axis at the point (x,y).

Call the horizontal projection of Δs Δx.

Since $\Delta s=\dfrac{\Delta x}{cos\,\varphi}$,

the equations (1) may be written, on multiplying both numerator and denominator by I_c:

$$\frac{1}{EI_c}\Sigma M\left(\frac{I_c}{I\,cos\,\varphi}\right)\Delta x=0,$$

$$\frac{1}{EI_c}\Sigma Mx\left(\frac{I_c}{I\,cos\,\varphi}\right)\Delta x=0,$$

$$\frac{1}{EI_c}\Sigma My\left(\frac{I_c}{I\,cos\,\varphi}\right)\Delta x=0.$$

Let $H=$ the horizontal thrust of the arch, and $t=$ the vertical intercept between the arch axis and the equilibrium polygon.

Then, evidently

$$M=Ht.$$

For the abbreviation, put

$$\left(\frac{I_c}{I\,cos\,\varphi}\right)\Delta x=w.$$

Inserting these values and placing constant factors outside of the sign of summation, we have

$$\Sigma wt=0,$$

$$\Sigma wxt=0,$$

$$\Sigma wyt=0.$$

In the second equation, x′, the horizontal projection of x, may be used in place of x, because the constant factor introduced thereby, can be put outside of the summation sign.

Again, since

$$t=(z-y),$$

where z represents the ordinate of the equilibrium curve, the last equations become

$$\left.\begin{array}{l}\Sigma wz-\Sigma wy=0,\\ \Sigma wx'z-\Sigma wx'y=0,\\ \Sigma wyz-\Sigma wy^2=0.\end{array}\right\} \quad \cdots\cdots\cdots\cdots\cdots \quad (2)$$

These equations can be further simplified as follows:

Divide the arch axis into any number of segments, designate the intervals $\triangle x_1$ $\triangle x_2$ etc. and number the middle point of each segment of arch axis 1,2 etc. as shown on Fig. 2.

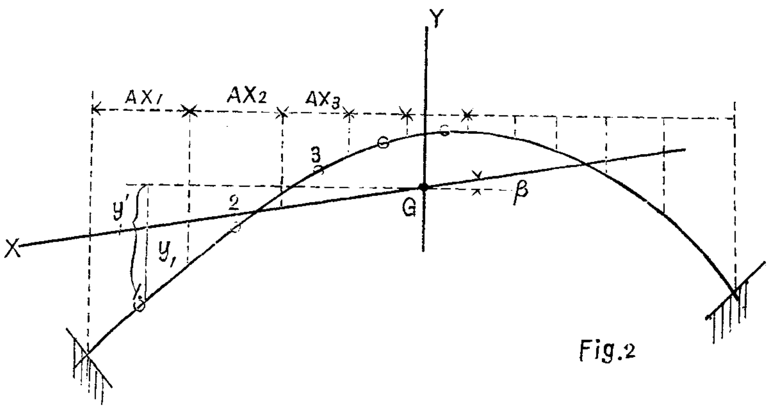

Fig.2

Let I_1,I_2 etc denote the moment of inertia at the point 1,2 etc. respectively and φ_1,φ_2 etc. the inclination of arch axis.

Compute the value of w for each segment:

$$w_1=\left(\frac{I_c}{I_1\cos\varphi_1}\right)\triangle x_1$$

$$w_2=\left(\frac{I_c}{I_2\cos\varphi_2}\right)\triangle x_2 \qquad \text{etc.}$$

It will be supposed that these quantities, so called "elastic forces", are concentrated at the corresponding points on the arch axis.

The center of gravity of these forces can be easily determined, either graphically or analytically.

If x–axis be taken through this center of gravity G, it is evident that, for any inclination of the axis to the horizontal, Σwy vanishes.

Moreover, if the inclination β of the same axis be so taken as to satisfy

$$tan\ \beta = -\frac{\Sigma wx'y'}{\Sigma wx'^2}, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (\ 3\ )$$

where y' denotes the vertical ordinate measured from the horizontal axis through G, then $\Sigma wx'y$ also vanishes.*

The demonstration is as follows:

Since $\qquad y = y' + x'\ tan\ \beta,$ we have

$$\Sigma wx'y = \Sigma wx'(y' + x'\ tan\ \beta)$$
$$= \Sigma wx'y' + tan\ \beta\ \Sigma wx'^2$$

On putting this equal to zero, we get the preceding value of β.

It will be observed, therefore, that, on locating the origin of co-ordinates at G and giving β the value found above, the equations (2) now reduce to

$$\left.\begin{array}{l} \Sigma wz = 0, \\ \Sigma wx'z = 0, \\ \Sigma wyz = \Sigma wy^2. \end{array}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (\ 4\ )$$

It may be remarked here that these conditions are true for any position of the vertical axis Y.

It is advisable however to take the axis also through the center of gravity G, so that

$$\Sigma wx' = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (\ 5\ )$$

Suppose now the w–forces are concentrated at the corresponding points on the equilibrium polygon instead of arch axis —— that is, at the intersections of verticals through 1,2 etc. to the equilibrium polygon.

Then, it is easy to see that, to satisfy the first condition $\Sigma wz = 0$, the center of gravity of these forces must lie on the X–axis.

This condition together with the remaining two enables us to draw the equilibrium polygon under any given loading.

It will be assumed that the loading for maximum moment of a certain section is known beforehand by means of influence lines or by reaction locus and tangent curves etc.

* Mehrten —— Statik der Bankonstruktionen. Bd. III. p.300

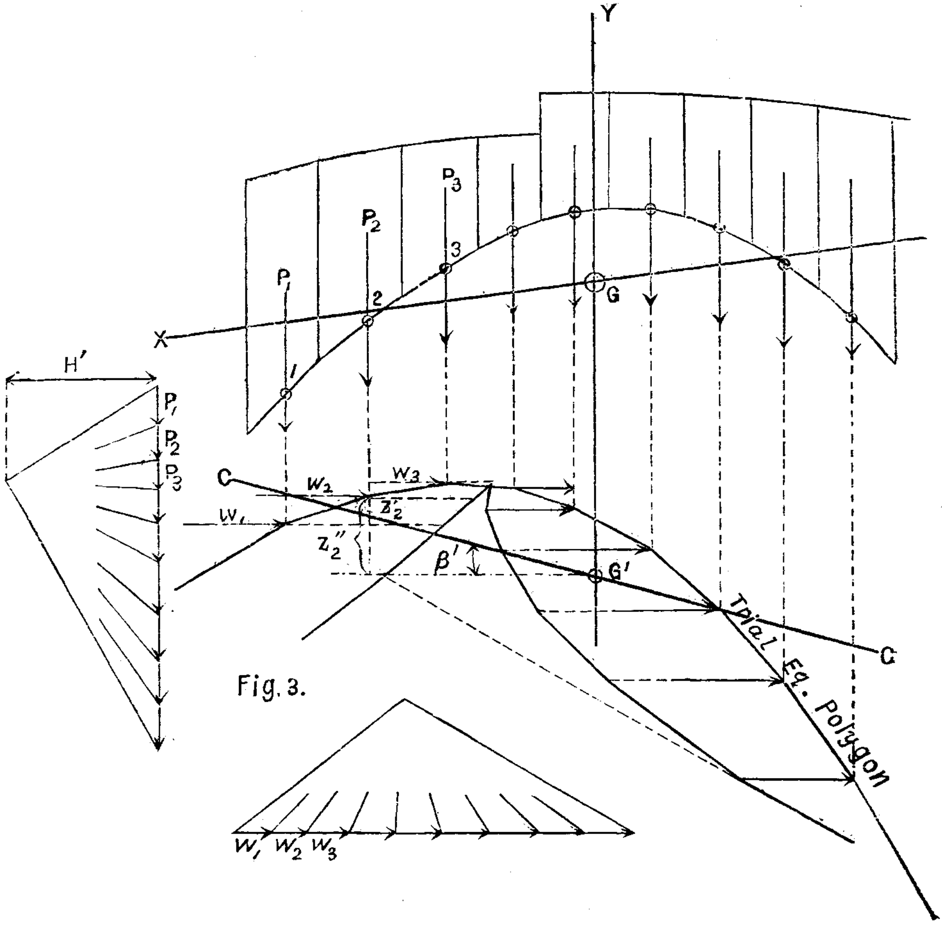
Y
P3
P2
3
P1
2
G
X
1
H'
P1
P2
P3
C
W2
W3
W1
Z'2
Z''2
β'
G'
Trial
Eq. Polygon
C
W1 W2 W3

Fig. 3.

Then the work proceeds as follows: Divide the arch axis at any interval, compute the w–forces, find the center of gravity and determine the co-ordinate axes X and Y. Compute the load on each segment, whence a trial equilibrium can be drawn with an assumed pole distance H'. (See Fig. 3)

Suppose now the w-forces situated on this trial equilibrium polygon and find the center of gravity G', either graphically or analytically.

It is to be remarked here that G' lies in the vertical through G, because G' must evidently lie in the vertical through the center of gravity of the w–forces concentrated on the true equilibrium polygon and further, that G is the center of gravity of the latter can be readily proved using the equation (5) and the condition $\Sigma wz = o$.

In Fig. 3 a graphical construction of finding G' is given, in which the w–forces are considered to act horizontally and the line of action of the resultant are produced to intersection with Y–axis.

We have now to establish the closing line. Any line passing through G' will satisfy the condition $\Sigma wz' = o$, where z' denote the ordinate of W on the trial equilibrium polygon. But the true closing line must also fulfill the second condition $\Sigma wx'z' = o$.

Evidently z' must be introduced for the trial equilibrium polygon in place of z in (4).

In Fig. 3, let z'' be the ordinate of the trial equilibrium polygon from the horizontal axis through G' and β' the inclination of the closing line C C to the horizontal. We have, then, in exactly the same way in which β was determined, from the last condition

$$\tan\beta' = \frac{\Sigma wx'z''}{\Sigma wx'^2} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \quad (6)$$

The closing line can be thus correctly established.

We have next to alter the coordinates z' in the ratio

$$\frac{\Sigma wy^2}{\Sigma wyz'},$$

and lay off the new ordinates vertically above or below the X–axis, according to sign.

This locates the true equilibrium polygon.

That the alteration of ordinates fulfills the third condition may be readily demonstrated:

Since by construction

$$z_1 = \frac{\Sigma wy^2}{\Sigma wyz'} z_1', \qquad z_2 = \frac{\Sigma wy^2}{\Sigma wyz'} z_2', \quad \cdots \cdots \cdots \cdots \cdots \cdots$$

We have, on multiplying both sides with corresponding factors $w_1 y_1$, $w_2 y_2$,······ and on adding them,

$$\Sigma wyz = \Sigma wy^2,$$

which is the third condition.

The subsequent mode of procedure, location of true pole for instance, does not differ essentially from that of usual method and any further treatment is unnecessary.

A word may be added as to the tabulation of quantities. After the arch axis is divided make out first Table I for the w – forces, then Table II to find β and β' and finally Table III for the ratio of alteration of the ordinates

Table I.

Points	Δx	I*	φ	$\cos\varphi$	$I \cos\varphi$	$w = \left(\frac{Ic}{I\cos\varphi}\right)\Delta x$
1						
2						
....						
....						

* For a reinforced concrete arch $I = I_c + nI_s$

Table II.

Points	x'	x'^2	y'	wx'	wx'	$wx'y'$	z''	$wx'z''$
1								
2								
....								
....								
					$\Sigma wx'^2$	$\Sigma wx'y'$		$\Sigma wx'z''$

Table III.

Points	y	z'	wy	wy^2	wyz'
1					
2					
....					
....					
				Σwy^2	$\Sigma wyz'$

In case the influence lines are used in determining the mode of loading for maximum moment etc., the tabulation should be much more simple, as most of the quantities should have been computed beforehand.

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