ON CASTIGLIANO'S THEOREM.

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If n forces or moments or both P_r, $r=1,2,\cdots n$, act on a body with the corresponding displacements or rotations δ_r, and if they are supposed to have attained their ultimate magnitudes by increasing uniformly from the initial magnitude zero, the work done on the body is

$$W=\frac{1}{2}\sum_{r=1}^{n}P_r\,\delta_r\cos(P_r,\,\delta_r).$$

If one of P's, say P_s, is supposed to be P_s+dP_s instead of P_s, we have

$$\frac{\partial W}{\partial P_s}=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{\partial P_r}{\partial P_s}\delta_r+\frac{\partial\delta_r}{\partial P_s}P_r\right)\cos(P_r,\delta_r).$$

If, on the other hand, at the loaded state of the body, we increase P_s uniformly by dP_s, we have

$$\frac{\partial W}{\partial P_s}=\sum_{r=1}^{n}\left(P_r+\frac{1}{2}\frac{\partial P_r}{\partial P_s}dP_s\right)\frac{\partial\delta_r}{\partial P_s}\cos(P_r,\delta_r)$$

$$=\sum_{r=1}^{n}\frac{\partial\delta_r}{\partial P_s}P_r\cos(P_r,\partial_r),$$

neglecting the infinitesimals of first order against the finite magnitudes.

Comparing these two results, we have

$$\frac{\partial W}{\partial P_s}=\sum_{r=1}^{n}\frac{\partial P_r}{\partial P_s}\delta_r\cos(P_r,\delta_r).$$

In an elastic solid τ of density ρ referred to three rectangular axes x, y, z, subjected to body forces $(K_x,\ K_y\ K_z)$ per unit mass, and with the surface o subjected to surface tractions $(T_x,\ T_y,\ T_z)$ per unit area, if $(X_x,\ X_y,\ X_z)$, $(Y_x,\ Y_y,\ Y_z)$, $(Z_x,\ Z_y,\ Z_z)$ are the stresses on the faces $dydz$, $dzdx$, $dxdy$ of a rectangular parallelopiped with its edges dx, dy, dz, parallel to the coordinate axes and with its angular point nearest to the coordniate origin at a point $(x,\ y,\ z,)$ in the interior of the body, and, if $(u_x,\ u_y,\ u_z,)$ are the displacements caused by the external forces, and $(\varepsilon_x,\ \varepsilon_y,\ \varepsilon_z)$, $(\sigma_x,\ \sigma_y,\ \sigma_z)$ are the rate of elongations and shears in the interior of the body, we have, denoting the time by t,

$$W=\int\rho\left[\left(K_x-\frac{\partial^2u_x}{\partial t^2}\right)u_x+\left(K_y-\frac{\partial^2u_y}{\partial t^2}\right)u_y+\left(K_z-\frac{\partial^2u_z}{\partial t^2}\right)u_z\right]d\tau$$

$$+\int (T_x u_x + T_y u_y + T_z u_z) do,$$

$$U=\int\int\int (X_x\varepsilon_x + Y_y\varepsilon_y + Z_z\varepsilon_z + Y_z\sigma_x + Z_x\sigma_y + X_y\sigma_z) dx dy dz,$$

U being the internal work done by the stresses. Making use of the equations

$$\frac{\partial X_x}{\partial x} + \frac{\partial Y_x}{\partial y} + \frac{\partial Z_x}{\partial z} + \rho\left(K_x - \frac{\partial^2 u_x}{\partial t^2}\right) = 0$$

$$\frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Z_y}{\partial z} + \rho\left(K_y - \frac{\partial^2 u_y}{\partial t^2}\right) = 0$$

$$\frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y} + \frac{\partial Z_z}{\partial z} + \rho\left(K_z - \frac{\partial^2 u_z}{\partial t^2}\right) = 0$$

$$Y_z = Z_y, \qquad Z_x = X_z, \qquad X_y = Y_x,$$

$$lX_x + mY_x + nZ_x = T_x$$

$$lX_y + mY_y + nZ_y = T_y$$

$$lX_z + mY_z + nZ_z = T_z$$

in which l, m, n are the direction cosines of the outward normal on do, and the tensions are taken positive, we shall have, by Gauss' integral theorem,

$$W = U.$$

Thus we arrive at the very important result

$$\frac{\partial W}{\partial P_s} = \sum_{r=1}^{n} \frac{\partial P_r}{\partial P_s} \delta_r \cos(P_r, \delta_r)$$

$$= \frac{\partial U}{\partial P_s} = \frac{\partial}{\partial P_s}\int\int\int (X_x\varepsilon_x + Y_y\varepsilon_y + Z_z\varepsilon_z + Y_z\sigma_x + Z_x\sigma_y + X_y\sigma_z) dx\, dy\, dz.$$

The current practice is to suppose P_r to be independent of P_s when $r \neq s$; but this is not necessarily the case.

Last formula also applies to statically determinate as well as indeterminate elastic systems.

If there are μ equations

$$F_\alpha(P_1, P_2, \cdots, P_n) = 0, \qquad \alpha = 1, 2, \cdots, \mu,$$

and if we take m of P's, $m+\mu=n$, as independent variables, we shall have

$$P_{t\alpha} = f_{t\alpha}(P_{s1}, P_{s2}, \cdots, P_{sm}), \qquad \alpha = 1, 2, \cdots, \mu,$$

so that

$$\frac{\partial W}{\partial P_{s\beta}} = \sum_{\alpha=1}^{\mu} \frac{\partial f_{t\alpha}}{\partial P_{s\beta}} \delta_{t\alpha} \cos(P_{t\alpha}, \delta_{t\alpha})$$

$$= \frac{\partial U}{\partial P_{s\beta}} = \frac{\partial}{\partial P_{s\beta}} \int\int\int (X_x\varepsilon_x + Y_y\varepsilon_y + Z_z\varepsilon_z + Y_z\sigma_x + Z_x\sigma_y + X_y\sigma_z)\, dxdydz,$$

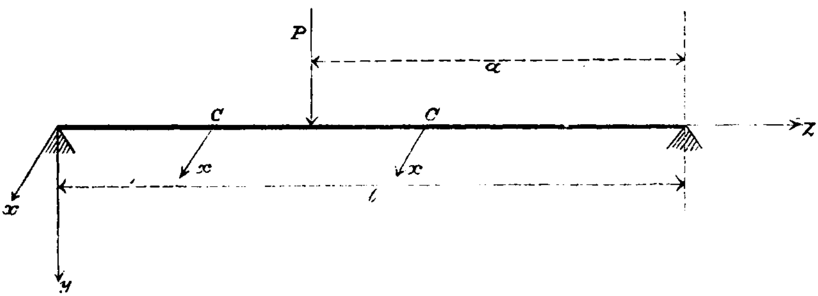
$\beta = 1, 2, \cdots, m,$

and these m equations, together with μ equations

$$F_\alpha = 0$$

may serve to determine all P's when δ's are given.

Ex. 1. To find the deflection δ at the loaded point of a simple beam of span l subjected to a single load P.



Here, if I is the moment of inertia of a normal cross section of the beam about x axis, and E the Young's modulus we easily see that

$$U = \frac{1}{2EI} \int_0^l M^2 dz$$

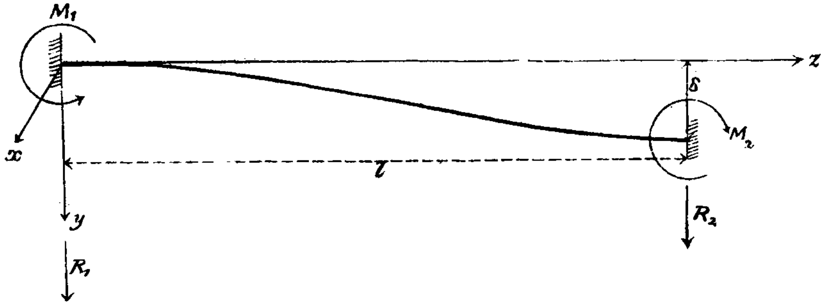
neglecting the work done by shears and denoting by M the bending moment about Cx. Thus

$$U = \frac{1}{2EI}\left[\int_0^{l-a} \left(\frac{Pa}{l} z\right)^2 dz + \int_{l-a}^{l} \left\{\frac{P(l-a)}{l}(l-z)\right\}^2 dz\right]$$

$$= \frac{P^2}{6EIl} a^2(l-a)^2.$$

$$\delta = \frac{\partial U}{\partial P} = \frac{P}{3EIl} a^2(l-a)^2.$$

Ex. 2. To find the moments and reactions of a beam of span l, whose one end is absolutely fixed and whose other end is fixed against the rotation and is displaced by δ normal to the beam.



Giving to I and E the same significations as before, and neglecting the work done by shears we have

$$U = \frac{1}{2EI}\int_0^l (M_1 + R_1 z)^2\,dz$$

$$= \frac{1}{2EI}\left(M_1^2 l + M_1 R_1 l^2 + \frac{R_1^2}{3} l^3\right),$$

and by the principle of Statics,

$$M_1 - M_2 + R_1 l = 0$$
$$R_1 + R_2 = 0.$$

If we take M_1 and R_1 as independent variables, we have

$$\frac{\partial U}{\partial R_1} = \frac{1}{2EI}\left(M_1 l^2 + \frac{2R_1}{3} l^3\right) = -\delta$$

$$\frac{\partial U}{\partial M_1} = \frac{1}{2EI}\left(2M_1 l + R_1 l^2\right) = 0,$$

whence it follows that

$$R_1 = -R_2 = -\frac{12EI\delta}{l^3}$$

$$M_1 = -M_2 + \frac{6EI\delta}{l^2}.$$

If we take M_1 and M_2 as independent variables, it will be convenient, although not necessary, to put U in the form

$$U = \frac{l}{6EI}\left(M_1^2 + M_1 M_2 + M_2^2\right).$$

Thus we have

$$\frac{\partial U}{\partial M_1} = \frac{l}{6EI}(2M_1 + M_2) = \frac{\delta}{l}$$

$$\frac{\partial U}{\partial M_2} = \frac{l}{6EI}(M_1 + 2M_2) = -\frac{\delta}{l},$$

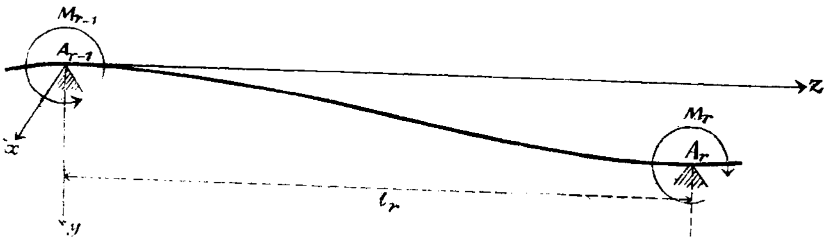
whence it follows that

$$M_1 = -M_2 = \frac{6EI\delta}{l^2}$$

$$R_1 = -R_2 = -\frac{12EI\delta}{l^3},$$

the same results as before.

Ex. 3. To find the theorem of three moments.



Let M denote the bending moment for rth span considered as a simple beam, then the reactions at A_{r-1} and A_r are

$$\frac{-M_{r-1} + M_r}{l_r} \quad \text{and} \quad \frac{M_{r-1} - M_r}{l_r}$$

respectively. Hence giving to I and E the same significations as before, and neglecting the work done by shears, we have

$$U_r = \frac{1}{2EI}\int_0^{l_r}\left(M_{r-1} + \frac{-M_{r-1} + M_r}{l_r}z + M\right)^2 dz$$

$$= \frac{l_r}{6EI}(M_{r-1}^2 + M_{r-1}M_r + M_r^2) + \frac{M_{r-1}}{EI}\int_0^{l_r} M\,dz$$

$$- \frac{M_{r-1} - M_r}{EIl_r}\int_0^{l_r} Mz\,dz + \frac{1}{2EI}\int_0^{l_r} M^2 dz.$$

Thus we obtain

$$\frac{\partial U_r}{\partial M_r} = \frac{l_r}{6EI}(M_{r-1}+2M_r) + \frac{1}{EIl_r}\int_0^{l_r} Mz\,dz = -\varphi_r + \frac{\delta_{r-1}-\delta_r}{l_r},$$

φ_r being the rotation at A_r.

Similarly for $(r+1)$th span, we have

$$U_{r+1} = \frac{l_{r+1}}{6EI}(M_r^2 + M_r M_{r+1} + M_{r+1}^2) + \frac{M_r}{EI}\int_0^{l_{r+1}} M\,dz$$

$$- \frac{M_r - M_{r+1}}{EIl_{r+1}}\int_0^{l_{r+1}} Mz\,dz + \frac{1}{2EI}\int_0^{l_{r+1}} M^2\,dz,$$

so that

$$\frac{\partial U_{r+1}}{\partial M_r} = \frac{l_{r+1}}{6EI}(2M_r + M_{r+1}) + \frac{1}{EIl_{r+1}}\int_0^{l_{r+1}} M(l_{r+1}-z)\,dz$$

$$= \varphi_r - \frac{\delta_r - \delta_{r+1}}{l_{r+1}}$$

Adding these partial derivatives of U_r and U_{r+1} with respect to M_r, we have, after multiplying the result by $6EI$ and transposing the integral terms,

$$l_r M_{r-1} + 2(l_r + l_{r+1})M_r + l_{r+1}M_{r+1}$$

$$= -\frac{6}{l_r}\int_0^{l_r} Mz\,dz - \frac{6}{l_{r+1}}\int_0^{l_{r+1}} M(l_{r+1}-z)\,dz$$

$$- 6EI\left(\frac{\delta_r - \delta_{r-1}}{l_r} + \frac{\delta_r - \delta_{r+1}}{l_{r+1}}\right),$$

a well-known theorem of three moments.

The case when I and E are not constant may be treated in an exactly similar manner.

It will be noticed that by finding the partial derivatives of U_r with respect to M_{r-1} and M_r, we shall have the solution of the general case of Ex. 2.