§ IX.

ON THE EQUILIBRIUM LIMIT IN GENERAL. PARTICULAR STUDY OF THE COLLAPSING STATE PRODUCED IN A PULVERULENT MASS AT THE MOMENT WHEN THE SUSTAINING WALL BEGINS TO OVERTURN.

42. *General Formulae of the Equilibrium Limit of the Isotropic Bodies which undergo Large Deformations.*

The study of a pulverulent mass at a dynamical state is accessible when it relates to the simplest cases only. In fact, in a sandy mass whose grains roll or slide one another with the notable relative velocities and suffer the displacements exceeding without limit their elastic limits, the stresses must have extremely complex values; because they probably depend on the actual elastic deformations of the couches as in the equilibrium state, and on the number of distinct molecular state passed over per unit of time, that is, on the relative velocity of sliding of the same couches as in the fluids, while the coefficients by which these velocities are there affected, instead of being constant, increases without doubt, in the same way as the coefficient of elasticity $\mu = mp$, with the mean stress p which measures the intimacy of contact of the contiguous particles.* The difficulty will also be equally very great with a plastic solid which we shall *knead* very rapidly, and when the stresses shall have also the dynamical parts as functions of the relative velocities of sliding.

But when, on the contrary, and that which almost always happens, the deformations are effected with sufficiently less rapidity that the inertia may be negligible and that the stresses exerted at each point do not differ sensi-

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* The motions of small amplitude, or *elastic*, which can be produced in the pulverulent masses seem to me to be less interesting, and I shall not be occupied with them. Their indefinite equations will be deduced from those of equilibrium (26), by simply deducting from X, Y, Z the components $-\frac{d^2u}{dt^2}$, $-\frac{d^2v}{dt^2}$, $-\frac{d^2w}{dt^2}$, per unit mass, of the inertia. These equations, in which N, T have the values (25), are not linear, even approximately, so long as the differences of the stresses exerted in various senses and at various instants are found to be comparable to these stresses themselves, as in the vibrating solids; it is then impossible to satisfy them by the expressions of u, v, w proportional to simple sines or cosines of linear functions of time. Thus the pulverulent medium cannot, at the natural state and under the influence of their elastic forces, execute the small *pendulous* motions; they suppress or transform the motions to another sort of vibrations emanated from the neighboring bodies and propagated there.

bly from the maximum elastic forces it becomes easy to establish the differential equations of the limiting equilibrium thus produced, provided that the body, solid or pulverulent, is and remains to be isotropic at the natural state.

It is important to observe that the total deformations suffered, up to the epoch t, by a particle of matter of very small dimensions in all directions, are then composed of two very distinct parts: these are, on one part, the deformations, called *inelastic*, *persistent*, *plastic*, *etc*, which will subsist if the particle becomes, at the epoch t, isolated from the rest of the body and abandoned to itself so as to be no more subjected to any external or internal pressure*; on the other, the small *elastic deformations* to which is due its actual state of tension or compression. We can consequently consider at each instant, *for every material volume element in particular*: 1° the *actual positions of natural state* (x, y, z) of its various points; 2° the *small elastic displacements* which separate these positions of natural state from the true positions. But, in general, the coordinates x, y, z under question do not vary with continuity when we pass from the matter of a particle to that of its neighbouring particles; because nothing is known that the various elements of volume, if we isolate them by abandoning each of them to itself, after we placed them against one another, can be in juxtapositions or are in perfect union. Thus, in the actual physical state of bodies, the positions of natural state, which are infinitely near for any two infinitely near material points or whose coordinates can serve as independent variables, do not in general exist.

The indefinite equations (26) of equilibrium (p. XXVI) have then the signification, only when x, y, z denote, not the coordinates of natural state, but the true actual coordinates of various particles, as I shall suppose to be in this paragraph. With this condition they remain to be applicable; and we know also that it will be sufficient, to make them exact, to join in their first members, to X, Y, Z, the components, with signs changed, of the small actual acceleration of the volume element. The formulae (24) or (25) [p. XXII or XXIII] of the forces N, T equally subsist if ∂, g denote

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* We cannot in general, *for a body of finite extent*, deduce, from the nullity of the external pressure, that of the internal pressures: but we can do it for a simple element of volume, because the suppression of the actions exerted on its surface involves the evanescence of the six quantities N, T, which have the values sensibly equal over all its extent; this evanescence takes place in itself by suitably making vary the six respective lengths and inclinations (which depend on N, T) of three material lines intersecting at one of its points.

the effective elastic deformations; but the expressions (15), (16) [p. XX] are admissible only when there are the actual coordinates x, y, z of natural state, that is so long as we are bound to employ them for simple volume elements in the interior of which we will suppose these expressions to be constant, without differentiating them with respect to x, y, z, or without taking in consequence, between N, T the relations similar to (28^{bis}) or (28^{ter}) [p. XXVIII].

The elastic deformations of a parallelopipedal volume element remain to be very small at every epoch and also vary gradually as its total deformations susceptible, on the contrary, to attain great values: their variations during an instant dt, or simply the increments received, during this instant, by the unit of actual length of its three edges and by the cosines of their respective angles, are thus sensibly reduced to the six small persistent deformations produced on the element during the same time dt. Suppose that the three edges under consideration are, at the epoch t, parallel to the axes of x, y, z; on one hand, the actual elastic deformations of the volume element shall be the six quantities ∂, g; on the other, if u, v, w denote, as in the treatise of hydrodynamics, the three components, at the epoch t, of the velocity at any point x, y, z, so that udt, vdt, wdt are the small displacements suffered at the end of the time dt, the six increments to be now considered will respectively be, after the formulae (15) and (16) [p. XX],

$$\frac{d\mathrm{u}}{dx}dt,\ \frac{d\mathrm{v}}{dy}dt,\ \frac{d\mathrm{w}}{dt}dt,\ \left(\frac{d\mathrm{v}}{dz}+\frac{d\mathrm{w}}{dy}\right)dt,\ \left(\frac{d\mathrm{w}}{dx}+\frac{d\mathrm{u}}{dz}\right)dt,\ \left(\frac{d\mathrm{u}}{dy}+\frac{d\mathrm{v}}{dx}\right)dt.$$

We see that it is permissible to regard them to be confounded with the persistent deformations produced during the time dt. Now, if a, b, c denote the cosines of the angles which any material rectilinear element starting from the point (x, y, z) makes with the axes, we find easily, by the application of a known formula (not differing from that at the bottom of the page XX whose first member is $\frac{du}{dx}$), that its elastic dilatation and its persistent dilatation for the instant dt, referred to the unit of time, have the respective expressions :

$$a^2\partial_x+b^2\partial_y+c^2\partial_z+bcg_{yz}+cag_{zx}+abg_{xy},$$

$$a^2\frac{d\mathrm{u}}{dx}+b^2\frac{d\mathrm{v}}{dy}+c^2\frac{d\mathrm{w}}{dz}+bc\left(\frac{d\mathrm{v}}{dz}+\frac{d\mathrm{w}}{dy}\right)+ca\left(\frac{d\mathrm{w}}{dx}+\frac{d\mathrm{u}}{dz}\right)+ab\left(\frac{d\mathrm{u}}{dy}+\frac{d\mathrm{v}}{dx}\right).$$

Now a single glance on the matter shows that, of all the rectilinear fibres which intersect at a point, the most *strained* are also, in an isotropic body, those which suffer the greatest *persistent dilatations*. It is then natural to suppose their ratio to the elastic dilatations,

$$(a)\quad \frac{a^2\frac{du}{dx}+b^2\frac{dv}{dy}+c^2\frac{dw}{dz}+bc\left(\frac{dv}{dz}+\frac{dw}{dy}\right)+ca\left(\frac{dw}{dx}+\frac{du}{dz}\right)+ab\left(\frac{du}{dy}+\frac{dv}{dx}\right)}{a^2\partial_x+b^2\partial_y+c^2\partial_z+bcg_{yz}+cag_{zx}+abg_{xy}},$$

to be positive and independent of the cosines a, b c which fix the direction of the fibre under consideration. By putting $b=0$, $c=0$; or $c=0$, $a=0$, or $a=0$, $b=0$, we find that this ratio has for its value some one of the three fractions

$$\frac{1}{\partial_x}\frac{du}{dx},\quad \frac{1}{\partial_y}\frac{dv}{dy},\quad \frac{1}{\partial_z}\frac{dw}{dz},$$

and that it can, in consequence, be reduced to

$$\frac{bc\left(\frac{dv}{dz}+\frac{dw}{dy}\right)+ca\left(\frac{dw}{dx}+\frac{du}{dz}\right)+ab\left(\frac{du}{dy}+\frac{dv}{dx}\right)}{bcg_{yz}+cag_{zx}+abg_{xy}}.$$

But if we make then $a=0$, or $b=0$, or $c=0$, we see that it is also equal to some one of the fractions

$$\frac{1}{g_{yz}}\left(\frac{dv}{dz}+\frac{dw}{dy}\right),\quad \frac{1}{g_{zx}}\left(\frac{dw}{dx}+\frac{du}{dz}\right),\quad \frac{1}{g_{xy}}\left(\frac{du}{dy}+\frac{dv}{dx}\right).$$

To admit, as an *expression of a fundamental physical law*, the invariability of the ratio (a) for all the rectilinear elements which intersect at a same point, it comes then to suppose the six actual elastic deformations ∂, g to be proportional to the six velocities corresponding to dilatation or sliding

$$\frac{du}{dx},\quad \frac{dv}{dy},\quad \frac{dw}{dz},\quad \frac{dv}{dz}+\frac{dw}{dy},\quad \frac{dw}{dx}+\frac{du}{dz},\quad \frac{du}{dy}+\frac{dv}{dx},$$

or, what comes to the same thing, to lay down the quintuple continued equality

$$(a')\quad \frac{\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}}{\partial_x+\partial_y+\partial_z}=\frac{\frac{du}{dx}-\frac{dv}{dy}}{\partial_x-\partial_y}=\frac{\frac{dv}{dy}-\frac{dw}{dz}}{\partial_y-\partial_z}=\frac{\frac{dv}{dz}+\frac{dw}{dy}}{g_{yz}}=\frac{\frac{dw}{dx}+\frac{du}{dz}}{g_{zx}}=\frac{\frac{du}{dy}+\frac{dv}{dx}}{g_{xy}}.$$

After the formulae (24) (with $A=0$) and (25) [p. XXII and XXIII], the denominators of these six fractions (a') are, if the body is pulverulent, in the same ratios as

$$0,\ \tfrac{1}{2}(N_1-N_2),\ \tfrac{1}{2}(N_2-N_3),\ T_1,\ T_2,\ T_3,$$

or, if it is solid, in the same ratios as

$$\frac{\mu}{5\lambda+2\mu}(N_1+N_2+N_3),\ \tfrac{1}{2}(N_1-N_2),\ \tfrac{1}{2}(N_2-N_3),\ T_1,\ T_2,\ T_3.$$

We then can, from the five relations (α') eliminate ∂, g, so as to make remain only N, T and the derivatives of the velocities u, v, w with respect to x, y, z. If, in particular, the body is supposed to be much less compressible than deformable, or if we have sensibly $\partial_x+\partial_y+\partial_z=0$, we have

$$(\beta)\begin{cases} \dfrac{du}{dx}+\dfrac{dv}{dy}+\dfrac{dw}{dz}=0, \\ \dfrac{2\left(\dfrac{du}{dx}-\dfrac{dv}{dy}\right)}{N_1-N_2}=\dfrac{2\left(\dfrac{dv}{dy}-\dfrac{dw}{dz}\right)}{N_2-N_3}=\dfrac{\dfrac{dv}{dz}+\dfrac{dw}{dy}}{T_1}=\dfrac{\dfrac{dw}{dx}+\dfrac{du}{dz}}{T_2}=\dfrac{\dfrac{du}{dy}+\dfrac{dv}{dx}}{T_3}. \end{cases}$$

Mr. de Saint Venant had arrived to the last four relations (β) by admitting that the stress exerted on each plane element has not the tangential component in the direction along which there is not, during the instant dt, the mutual sliding of the couches parallel to the plane element.

We have already, among the six unknown forces N, T and the three components u, v, w of the velocity, the eight indefinite equations (26) [p. XXVI] and (β). It remains to find one last indefinite relation. This will be exactly the characteristic equation of the plastic or collapsing state, which expresses that the deformations ∂, g attain at each instant the most remote elastic limits permitted by the substance and the mode of distribution of the stresses employed. If ∂_1, ∂_2, ∂_3 denote the three principal elastic dilatations at some point (x, y, z), their maximum difference $\partial_1-\partial_3$ in the equilibrium limit, acquires, for each given value of the cubical dilatation $\partial_1+\partial_2+\partial_3$ and of the ratio $(\partial_1-\partial_2)/(\partial_2-\partial_3)$ of the differences of the middle dilatation ∂_2 to the two extremes, a determinate value, which is the greater as the elastic limits are greater. By calling then a certain positive function f, we shall have,

$$(\gamma)\qquad \partial_1-\partial_3=f\left(\partial_1+\partial_2+\partial_3,\ \frac{\partial_1-\partial_2}{\partial_2-\partial_3}\right),$$

or, in the case of a sensibly incompressible body for which $\partial_1+\partial_2+\partial_3=0$,

$$(\gamma') \qquad \partial_1 - \partial_3 = f\left(\frac{\partial_1 - \partial_2}{\partial_2 - \partial_3}\right) = f\left(\frac{\partial_1 - \partial_2}{\partial_1 + 2\partial_2}\right) = f\left(\frac{-2\partial_2 - \partial^3}{\partial_2 - \partial_3}\right).$$

We shall substitute in (γ) or (γ'), for ∂_1, ∂_2, ∂_3, their values resulting from the formulae (5) or (10) and containing F_1, F_2, F_3, then we shall suppose these principal elastic forces F to be evaluated as functions of N, T, so as to transform the equation (γ) or (γ') to a relation under finite form between the six stresses N, T. The coefficient of elasticity λ, μ or m which there appear can be regarded to be constant; because it is natural to admit, and the experience proves, that they remain almost the same in a body which we deform without diminishing or increasing sensibly its density.

If, for example, we consider a plastic solid, the formula (γ') becomes

$$(\gamma'') \qquad F_1 - F_3 = 2\mu f\left(\frac{\partial_1 - \partial_2}{\partial_2 - \partial_3}\right) = 2\mu f\left(\frac{F_1 - F_2}{F_2 - F_3}\right).$$

In the problems of plane deformations, in that of the torsion of a circular cylinder, etc., we have $\partial_2 = 0$, $\partial_3 = -\partial_1$, and this formula is reduced to $F_1 - F_3 =$ a constant $2\mu f(1)$ or $2K$, as we have seen at Art. 26 (p. LIII). In the equally simple questions of the extension, of the compression, and of a circular flexure of a prism, the middle dilatation ∂_2 is equal, by the reason of symmetry, to the smallest dilatation ∂_3 for the extended fibres, and to the greatest ∂_1 for the contracted fibres. The formula (γ'') gives then to the difference $F_1 - F_3$ of the extreme elastic forces the respective constant values $2\mu f(\infty)$, $2\mu f(0)$, which can differ from each other and also from the value $2\mu f(1)$ relating to the case $\partial_2 = 0$. Thus, in the especially important problems to be now considered, we shall have for the equation special to the limiting equilibrium $F_1 - F_3 =$ a constant $2K$; but the quantity K will not probably be completely the same for the three cases. Nevertheless, the experiences of Mr. Tresca tend to show that we can, without great error, put simply $F_1 - F_3 =$ a constant at the interior of any homogeneous solid body at the plastic state, at least when the body is much more deformable than compressible.

Besides the indefinite equations, there will be the conditions special to the surface of the body. They consist: 1° for the points where the external pressure will be known, in equating the respective component of the forces sustained by the two faces of a superficial couch; 2° against a fixed wall, in supposing the velocity to be in the same sense as the tangential component of the pressure exerted on the element of contiguous wall, and in equating to a constant coefficient of external friction the ratio of this tangential component to the normal component of the pressure; 3° for other points, in giving them at each epoch the effective components u, v, w of the velocity. These last conditions will be absolutely necessary to the calculation of the absolute magnitudes of u, v, w, which the indefinite equations (α') or (β) determine *at the most* the ratios at various points.

Lastly, the body generally remains at the elastic or stable state in a more or less large region. We obtain the equation of the variable surface which separates this region from one where the persistent deformations are produced, by expressing that the elastic limit just begins to be attained in the portion where the contexture does not change. It must be remarked that the deformations, persistent or elastic, vary with continuity in all the extent of the body whose state is transformed gradually from a point to the neighbouring points, provided there does not occur the rupture: only, the former are insensible, or at least very nearly invariable from an instant to the other, in the portion where the molecular constitution is stable, while the latter attain, in the other portion, the most distant elastic limits permitted by the substance and the modes of deformation employed. I neglect, for the sake of simplicity, the third intermediate region, probably of a small extent in soft bodies, and in which the matter, at the state called one of *imperfect elasticity*, is in the way to become *springy*, that is to enlarge its elastic limits immediately attained yet susceptible to deviate.

We can observe in a memoir of Mr. de Saint Venant in the volume of 1871 of the *Journal de Mathématiques de M. Liouville*,* how the above formulae lead easily to the laws of the torsion of a circular cylinder and to those of the uniform flexure of a prism, when the deformations surpass the elastic limits.

42.^{bis} *Constancy of the Flowing Velocity of the Sand from an Orifice.*

The indefinite equation (γ') presents a remarkable specialty in the case of a pulverulent mass that, after the formulae (14) [p. XVII] the differences $\partial_1 - \partial_3$, $\partial_1 - \partial_2$, $\partial_2 - \partial_3$ vary with only the mutual ratios of the stresses N, T; thus this equation, similar to the five indefinite relations (β) and to the

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* Complement to the preceding memoirs, etc. — See also, of the same author: 1° an article inserted in the *Comptes rendus* de l'Académie des science de Paris (t LXXIV, 15 April 1872), on a very remarkable particular case of plane deformations, the case of a cylindrical ring whose fibres parallel to the axis deviate from this axis, symmetrically all round, by preserving their parallelism and height; 2° another article of 20th Nov. 1871 (t. LXXIII) on the torsion of a circular cylinder (where a note, relative to the *untwisting* which will be produced if we leave a twisted cylinder to itself, seems to me to be modified, by the reason that there no account is taken of the actual state of *maximum* elastic tension of the couches in limiting equilibrium).

conditions concerning the limiting surface of the mass in state of collapse, does not cease to be satisfied when, for same values of u, v, w, we make N, T vary in any constant ratio. Consequently, if the portion (of the mass) where the collapsing is produced has a weight sufficiently small, in comparison with the difference of the stresses which it sustains in opposite senses, that we can suppress in the three indefinite equations (26) [p. XXVI] the terms ρX, ρY, ρZ, or make these equations homogeneous, as the other, in N, T, the stresses can vary in any one ratio without ceasing to make an equilibrium and without that nothing is changed as to the velocities u, v, w.

Conceive, for instance, a reservoir pierced at its bottom with an orifice which is sufficiently small that most part of a pulverulent mass which we lead out there is unmoved, and admit moreover that this mass has a coefficient of internal friction sufficiently great that, under the moderate loads, the accelerations of the other part are at the most comparable to the gravity g. It is clear that the mean stress p, zero at the orifice, will increase rapidly as we advance thence toward the interior; in consequence, the weight of the mass in motion and its inertia will be negligible, in the formulae (26), in comparison with the derivatives of N, T. These equations, thus simplified, joined to the other equations of the problem, determine, for the various points of the portion of the reservoir where the sand flows, the values of the stresses N, T which mutually equilibrate, each vanishing at the orifice; besides, however great the load may be, these stresses preserve among them their ratios, and the same values of u, v, w do not cease to satisfy the equtions under consideration. Consequently, when the height of load is much greater than the dimension of the orifice, the stresses which thereby result are sensibly neutralised by the friction, and the flow is not produced except under the influence of very weak causes, neglected in our analysis. Also these causes do not increase indefinitely with the load: because the stresses N, T, in the region where the velocities are sensible, make equilibrium as to their principal or increasing part with the load; they cannot contribute in producing the accelerations and consequently the velocities u, v, w, concurrent with the gravity, which, with their neglected parts, do not cease to be comparable with the weight of the matter in motion. *In a flow of sand from an orifice, the velocity thus tends toward a limit as the height of load becomes a little great, and it keeps itself constant afterward.* Thus is explained the uniformity of the flowing which the ancients obtained with the hour glass which served them to measure the time*

43. *The State of Collapse is Established All at Once in a Notable Portion of the Masses.—Its Differential Equations.*

Under the particular hypothesis of plane deformations, to which we are most frequently referred when we treat the pressure of earth, the components T_1 T_2 N_3 are respectively 0, 0, $-p$, as we have seen at page XXVI (and as will also appear from the formulae (β) where we shall make $w=0$, $du/dz=0$, $dv/dz=0$): thus it is sufficient to join to the two indefinite equations (28) [p. XXVII] the characteristic relation of the equilibrium limit, as well as the definite conditions in which N, T appear, to have all the formulae necessary for the determination of the stresses. We can then dispense with the calculations of the velocities u, v, and this I shall do in the following Articles.**

I will consider at first a heavy sandy mass whose top surface shall be plane, and I will suppose that, in case it is shaken by virtue of a commencement of overturning of the wall which it sustains, it collapses with less velocities.

At such a moment, is the state of collapse established along a simple horizontal line parallel to the intersection of the wall and of the top slope, or rather on all the extent of a *surface of rupture*, geometrical situation of an infinity of similar straight lines, or finally does it attain almost instantaneously a final volume of the matter of the mass?

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* We see that this fact is by no means the indication of a pretended impossibility of a sandy mass, compressed in certain directions, to transmit in the perpendicular directions a sensible fraction of the stresses which it sustains, as has been believed to be capable to infer it by Mr. Beaudemoulin, ancien ingénieur en chef des ponts et chaussées, in a work (*Études sur une propriété spéciale du sable et sur ses applications*) printed in the recueil des *Mémoires de la Société des ingénieurs civils* (Paris, 1874). If, in conformity to the thesis in this work, the sand had been absolutely deprived of elasticity, there will not be the *pressure* of earth, and it will be sufficient, to sustain a vertically cut mass, to cover it by a light coat which obstruct the superficial particles from detaching. The apparatus of removing the centres, by the sand, of the honourable engineer constitute nevertheless an invention as ingeneous as useful.

** The calculation of the velocities u, v seems ought to be much more difficult. We can see, in an article of the *Comptes rendus* (t. LXXIV, 12 February 1872), what partial differential equation of the second order, linear but with variable coefficients, must be integrated to determine them. In this equation the independent variables are the orthogonal curvilinear coordinates defined by the two families of *isostatic* (or better *orthostatic*) cylinders produced in the medium. These cylinders, on all the extent on which the stresses are exerted are normal, play in themselves, in a medium, at the plastic or collapsing state, subjected to the pressures much exceeding its weight, some interesting properties. I have studied them in three other articles inserted in the *Comptes rendus* (22 and 29 January 1872, t. LXXIV, and 22 September 1873, t. LXXVII).

If the mass had been solid and a *corner* of matter tends to be detached, the rupture will be produced at first along the horizontal straight line, perpendicular to the planes of deformations, on which will be found the *critical point* relative to each of these planes, that is the point where the principal positive dilatation ∂_1 will attain its greatest value. It will be propagated whence to another parallel and adjacent straight line, comprising the series of the critical points which will correspond to the next state of the mass. From place to place, this will be found to be divided into two, along a cylindrical surface of rupture, without ever being able in any manner to utilise for its defense all the means of resistance to the destruction possessed by its various parts. In fact, it will be only at the critical points, forming at each instant a material line of an insensible length and thickness (or at most a surface in the case of a mass which will slide in block on an underlying couch of weak cohesion parallel to the top slope), that the *tension,* the effort opposing to the separation of the parts, will attain its limiting value. The equilibrium state which we wish to study will consequently exist, at any moment, only in an infinitely small extent.

But the case will not be the such; because the particles of the pulverulent media enjoy a mobility not possessed by the solid bodies, and it is natural to admit that the difficulty less than they undergo to displace one against the other permits these media to resist in a measure more nearly equal to the sort of rupture which they present when they collapse. In fact, in all the stable equilibrium modes considered in the preceding paragraphs, the deformations ∂_1 are constant at various points of the mass, such that *these points become critical all at once.* We can then admit that, *when a wall which sustains the earth without cohesion begins to overturn, the equilibrium limit is established almost immediately up to a sufficiently great distance behind its back face,* by leaving at most only the limited regions of the mass, as, for example, a couch of more or less thickness contiguous to the wall and protected by its friction. The equations themselves of the equilibrium limit indicate in which case a certain portion of earth adjacent to the back face of the wall shall be thus preserved at the commencement of the failure.

These equations comprehend, as we see:

1° The two indefinite equations (28) [p. XXVII] which express the equilibrium of translation of a rectangular volume element;

2° A third indefinite equation, signifying that at all the points of the mass the elastic limit is attained, or, what comes to the same thing (p. LI), that the maximum inclination of the stresses to the respective normals to the plane elements which they urge is at each point equal to the angle ϕ

of internal friction; after the formula (66^{bis}) [p. L], taking an equality and denoting by R the radical (31) [p. XXVIII], this relation is nothing but

$$(93)\quad \frac{1}{p}\sqrt{T^2+\left(\frac{N_2-N_1}{2}\right)^2}=sin\phi, \text{ or } \sqrt{T^2+\left(\frac{N_2-N_1}{2}\right)^2}=-\frac{N_1+N_2}{2}sin\phi;$$

raised to the second power, it takes the form which Macquorn Rankine has given it

$$(94)\qquad 4T^2+(N_2-N_1)^2-(N_1+N_2)^2sin^2\phi=0;$$

3° Lastly the special conditions, either at the free surface or top slope, or at the surface of separation of the mass and the sustaining wall. The former amounts to that the two components, normal and tangential, of the stress exerted by the mass on its surface couch, are zero at all the points of the free surface: these conditions, combined with the indefinite equation (94), must lead to

$$(95)\qquad N_1=0,\quad N_2=0,\quad T=0\quad \text{(at the free surface)}.$$

Latter relation, special at the back face of the wall, is applicable only when the contiguous particles of the mass are about to undergo finite slidings, a circumstance which seems ought to be produced since the commencement of overturning of the wall, as often as it will not be in contradiction with the other equations of the problem. Now its realization requires that the angle made at each point, with the prolongation of the normal to the back face of the wall, by the pressure which is applied to it, is exactly equal to the angle of the maximum friction of the wall and of the sandy matter of the mass.

The introduction of this last condition in the new theory is due to Mr. Maurice Levy.*

Macquorn- Rankine, in his memoir *On the stability of loose Earth***, in the PHILOSOPHICAL TRANSACTIONS of London (1856—1857), assimilates a mass limited by a wall to an indefinite mass; he contents to express, by virtue of the hypothesis on the maximum frictions at each point at the moment when a collapsing begins, that the weight of the mass is neutralized as far as it is possible by these frictions, and the pressure exerted on the wall is reduced in consequence to its minimum value when the collapsing tends to be produced downward, while the contrary will be the case and the

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* Poncelet had already employed it in the old (*Mémoire sur la stabilité des revêtements*, n° 138; in n° 13 of the *Mémorial de l'officier du génie*, 1840).

** Mr. Flamant, ingénieur des ponts et chaussées à Lille, is to publish a French translation in the *Annales des ponts et chaussées* (5ᵉ série, t. VIII, 1874).

pressure will become maximum if the wall, instead of moving away, approaches the earth and compresses it.

44. *Integration of These Equations, when the Back Face of the Wall has a Certain Inclination to the Vertical, or when the Angle of External Friction has a Certain Value.*

All these conditions, with the exception of the last, relating to the wall are found evidently verified by the two solutions which we have previously studied in Arts. 33,34 (pp. LXIV to LXVIII), and which are presented to us to correspond to two extreme cases of the ordinary equilibrium or of elasticity of a mass without cohesion. Now, if

denote the inclination of the sustaining wall to the vertical, each of these solutions will give, by putting

$$\varepsilon_1 = i,$$

a certain value ϕ_1 [second formula (77)] for the angle made, with the prolongation of the normal to the plane elements which have precisely the direction of the wall, by the pressure exerted on these plane elements, and it can lead to the result that ϕ_1 is just equal to the angle of mutual friction of the wall and the mass. Admit that it may be the such: then all the conditions of the equilibrium limit shall be verified by the solution under consideration, and it will be admissible, provided it is besides that which gives the minimum values of the pressures, in the ordinary case relating to a limiting equilibrium corresponding to a *collapsing by pulling*, or, on the contrary, that which gives the maximum values relating to an equilibrium limit corresponding to a *collapsing by compression.*

Confining our attentions to the first solution, the most interesting one in practice, we obtain the results given by Rankine and by Mr. Levy. They are resumed in the formulae (77), if we adopt for ψ the smallest, in absolute value, of the roots given by the first of these formulae. In particular, the two components, normal $-\mathfrak{N}$ and tangential $\mathfrak{T}$, of the pressure exerted per unit area at various points of the wall, will be obtained by putting $\varepsilon_1 = i$. As to their resultant $\mathfrak{R}$, it will be equal to $\mathfrak{T}/sin\phi_1$. It will be preferable to replace, as in the Art. 35 (p. LXIX), the distance l normal to the top slope by the oblique distance L, measured along the wall itself, and such that $l = Lcos(\omega - i)$. We shall then compose all the elementary pressures to a single total pressure P, as we have done at Art. 35, and it will give finally, instead of the formulae (82):

$$(96)\quad \begin{cases} L_1 = \frac{2}{3}L, \quad F = K\dfrac{\rho g L^2}{2}, \\ \text{with } sin(\omega + 2\psi) = \dfrac{sin\omega}{sin\phi}, \quad tg(\phi_1 + i + \psi) = \dfrac{tg(i+\psi)}{tg^2(\frac{\pi}{4} - \frac{\phi}{2})}, \\ K = \dfrac{sin\phi}{2cos^2(\frac{\pi}{4} - \frac{\phi}{2})} \dfrac{cos(\omega - i)cos\psi\, sin2(i+\psi)}{cos(\omega + \psi) sin\phi_1}. \end{cases}$$

Thus, *when a sustaining wall begins to overturn and the angle of the external friction has precisely the value ϕ_1 resulting from the fourth equation of* (96), *the state of collapse is established in all the extent of the mass as soon as the last elastic equilibrium mode on all this extent ceases to exist.* The reason is that one and the same distribution of the stresses suits to these two states, which can follow each other without discontinuity. On the contrary, when the angle of external friction is greater than the root ϕ_1 of the equation under consideration (96), the inclination of the pressure to the prolongation of the normal to the back face of the wall is found too weak that the state of collapse is produced, in the region contiguous to the wall, since the instant at which the last elastic equilibrium common to all the mass disappears. It must then happen that a corner of matter adjacent to a wall remains constantly at the elastic state, at least during the initial period of the overturning during which the velocities are yet insensible, or the state of collapse propagates rapidly in this region, so as to extend to all the mass at the end of a very short instant. Art. 47 in sequel contains the study of these interesting circumstances.

45. *Case in which the Angle of the External Friction is Equal to That of the Internal Friction ϕ.*

The walls which we construct effectively are always sufficiently rigorous not to disturb a thin couch of the mass which they sustain, and it is against this couch that the rest of the inconsistent mass can slide.

The angle of external friction is then equal to ϕ. In fact, when two contiguous couches of a pulverulent body are subjected to the actions, increasing gradually from instant to instant, which tend to make them slide against each other, the ruptures or the *finite* slidings are produced, after the experimental and common law of friction, along the plane elements for which the inclination of the stress which they sustain to their normal is maximum and attains a determinate value, which is the angle of mutual friction of earth on earth: now ϕ denotes precisely, as we have seen at Art. 25 (p. LI), the maximum inclination under question at the points where the rupture is imminent. Thus *the angle of mutual friction of earth on earth at the instant when a finite sliding tends to be effected, is just equal to the angle ϕ of internal friction or of sliding earth.*

Consequently, the preceding formulae are applicable only when the wall has exactly the direction of the plane elements for which the inclination of the stress which they sustain to the prolongation of their normal attains its maximum value ϕ.

We have seen at Art. 33 (p. LXVI) that the plane element subjected to the minimum stress makes the angle $-\psi$ with the vertical; after the law which concludes Art. 25 (p. LI), the inclination of the wall to this plane element must be $\frac{\pi}{4}-\frac{\phi}{2}$, which gives two directions, or two possible values, $-\psi\pm(\frac{\pi}{4}-\frac{\phi}{2})$, of i. In practice, the former will be the only one sufficiently small in absolute value to be admissible. We can then employ the formulae (96) in the calculations of limiting equilibrium, only when the inclination of the sustaining walls to the vertical will be

(97) $$i=\frac{\pi}{4}-\frac{\phi}{2}-\psi.$$

With this condition, the angle ϕ_1 will just attain its maximum value ϕ.

The expression (96) of K then becomes extremely simple. We can make at first

$$2\cos^2(\tfrac{\pi}{4}-\tfrac{\phi}{2})=1+\sin\phi,\quad \phi_1=\phi,$$

and, after (97),

(97^{bis}) $$2(i+\psi)=\frac{\pi}{2}-\phi;$$

which reduces it to

(98) $$K=\frac{\cos\phi\cos\psi\cos(\omega-i)}{(1+\sin\phi)\cos(\omega+\psi)}.$$

It is now easy to eliminate ω. The third equation (96), if we replace $\sin(\omega+2\psi)$ by $\sin\omega\cos2\omega+\cos\omega\sin2\psi$, gives

(98^{bis}) $$\frac{\cos\omega}{1-\sin\phi\cos2\psi}=\frac{\sin\omega}{\sin\phi\sin2\psi}:$$

in other words, the cosine and the sine of ω are proportional to $1-\sin\phi\cos2\psi$ and $\sin\phi\sin2\psi$. Consequently, $\cos(\omega-i)$, or $\cos\omega\cos i+\sin\omega\sin i$, will be proportional to

$$\cos i-\sin\phi(\cos i\cos2\psi-\sin i\sin2\psi)=\cos i-\sin\phi\cos[2(i+\psi)-i]$$
$$=\cos i-\sin\phi\cos[\tfrac{\pi}{2}-(\phi+i)]=\cos i-\sin\phi(\sin\phi\cos i+\cos\phi\sin i)$$
$$=\cos^2\phi\cos i-\cos\phi\sin\phi\sin i=\cos\phi\cos(\phi+i).$$

On the other hand, $\cos(\omega+\psi)$, or $\cos\omega\cos\psi-\sin\omega\sin\psi$, will equally be proportional to

$$\cos\psi - \sin\phi(\cos\psi\cos 2\psi + \sin\psi\sin 2\psi) = \cos\psi(1 - \sin\phi).$$

The quotient of $\cos(\omega - i)$ by $\cos(\omega + \psi)$ is then equal to

$$\frac{\cos\phi\cos(\phi + i)}{\cos\psi(1 - \sin\phi)},$$

and the formula (98) becomes

$$(99) \qquad K = \cos(\phi + i).$$

This formula (99) is due to Mr. Maurice Levy (see Arts. 18 and 19 of his Memoir). The preceding demonstration adds to the advantage of the simplicity that of deducing the particular value (99) from the more general expression (96) of K.

46. *Approximate Integration for the Walls of a Different Inclination.*

Mr. de Saint Venant has indicated an approximate method* to obtain the laws of the limiting equilibrium when the angle ϕ_1 of the external friction, instead of being that which is given by the fourth formula (96), differs from it by a sufficiently small but sensible quantity. This method consists in joining, to the expressions (77), then unsound, of $\mathfrak{T}$ and $-\mathfrak{N}$, the small complementary terms which must render them exact, and to substitute these total values in the equations of equilibrium, by neglecting, in (94), the products and the squares of the small complementary terms against their first powers. I have treated at his demand this problem in a note inserted in the *Compte rendu* of April 4th, 1870.** But there is a means much to simplify the calculations by changing the coordinate axes, as I here proceed.

I will choose the axis of y' along the straight line drawn through the origin O (fig. 5) and making with the vertical the angle $-\psi$, and which, in the case when the formulae (77) are applicable, is found to be just parallel to the plane elements subjected to the minimum stress, as we have seen at the end of Art. 33. Consequently, in the same case, the directions of the plane elements for which the inclination of the stress which they sustain to the prolongation of their normal attains its greatest value ϕ, are those of the two straight lines OQ, OQ', which respectively make with Oy' the angles $\frac{\pi}{4} - \frac{\phi}{2}$ and $\pi - (\frac{\pi}{4} - \frac{\phi}{2})$. The axis of x' will be the bisector of the angle QOQ' and will have $\frac{\pi}{2} - \psi$ for the inclination to the vertical OG.

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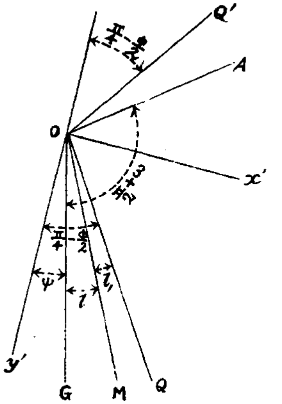
* See Art. 7 of his memoir inserted in the *Compte rendu*, of the session of February 14th, 1870 (t. LXX, p. 283).

** Tome LXX, p. 751.— See also, in the same number of the *Compte rendu*, p. 718, an article of Mr. de Saint Venant on the same subject.

Fig. 5



The components N_1', N_2', T', along the new axes, of the pressures (or rather tensions) exerted on the plane elements which shall be perpendicular to them, will be composed:

1° Of their parts corresponding to the special solution already studied, and which are $-(\rho g l \sin\psi)/\sin(\omega+\psi)$, $-(\rho g l \cos\psi)/\cos(\omega+\psi)$ [after (77^{ter})] for N_1', N_2', and zero for T';

2° Of small yet unknown parts $\rho g n_1$, $\rho g n_2$, $\rho g t$.

We shall have then

$$(100)\, N_1'=\rho g\left[-\frac{\sin\psi}{\sin(\omega+\psi)}l+n_1\right],\ N_2'=\rho g\left[-\frac{\cos\psi}{\cos(\omega+\psi)}l+n_2\right],\ T'=\rho g.$$

These values must be introduced in the indefinite equations of equilibrium relative to the new axes. We have, to obtain them,

1° Only to accent x, y, N_1, N_2, T in the formulae (28) [p. XXVII], applicable to each system of rectangular axes situated in the plane of xy, and to express that the inclination α of the gravity to the axis of y' is here equal to ψ; whence follows

$$(101)\qquad \frac{dN_1'}{dx'}+\frac{dT'}{dy'}+\rho g\sin\psi=0,\ \frac{dT'}{dx'}+\frac{dN_2'}{dy'}+\rho g\cos\psi=0;$$

2° Only to accent N_1, N_2, T in equation (94), equally the same for each system of rectangular axes parallel to the plane of deformations, and which becomes

(101[bis]) $$4T'^2+(N_2'+N_1')^2-(N_2'+N_1')^2 sin^2\phi=0.$$

Lastly, it must be observed that the perpendicular l to the top slope OA, being inclined to the vertical by ω and consequently to Oy' by $\omega+\psi$, has for its expression

(101[ter]) $$l=x' sin(\omega+\psi)+y' cos(\omega+\psi).$$

With this value of l, the expressions (100) of N_1', N_2', T', introduced in two equations (101), reduce them to

(102) $$\frac{dn_1}{dx'}+\frac{dt}{dy'}=0, \quad \frac{dt}{dx'}+\frac{dn_2}{dy'}=0.$$

The first of these new relations shows that n_1 and $-t$ are the two derivatives in y' and x' of a function ϖ_1 of x', y'; the second signifies similarly that n_2 and $-t$ are the two derivatives in x' and y' of a single function ϖ_2. We have thus

$$-t=\frac{d\varpi_1}{dx'}=\frac{d\varpi_2}{dy'},$$

and ϖ_1, ϖ_2 are the two derivatives in y' and x' of a single function ϖ. The two equations (102) are then equivalent to

(102[bis]) $$n_1=\frac{d^2\varpi}{dy'^2}, \quad n_2=\frac{d^2\varpi}{dx'^2}, \quad t=-\frac{d^2\varpi}{dx'dy'};$$

which reduce the determination of the three unknowns n_1, n_2, t to that of the unique function ϖ, or rather to that of its three second derivatives in x', y'.

There remains, to calculate ϖ, the indefinite equation (101[bis]), in which we must substitute for N_1', N_2', T' their expressions (100) by neglecting the squares and the products of the small quantities n_1, n_2, t. If we observe that

$$\tfrac{1}{2}(N_2'-N_1')=\rho g\left[-\frac{sin\omega}{sin2(\omega+\psi)}l+\frac{1}{2}\left(\frac{d^2\varpi}{dx'^2}-\frac{d^2\varpi}{dy'^2}\right)\right],$$

$$\tfrac{1}{2}(N_2'+N_1')=\rho g\left[-\frac{sin(\omega+2\psi)}{sin2(\omega+\psi)}l+\frac{1}{2}\left(\frac{d^2\varpi}{dx'^2}+\frac{d^2\varpi}{dy'^2}\right)\right],$$

and that $sin(\omega+2\psi)=(sin\omega)/sin\phi$ [by virtue of the first formula (77)], it must become

$$-l\frac{sin\omega}{sin\,2(\omega+\psi)}\left[(1-sin\phi)\frac{d^2\varpi}{dx'^2}-(1+sin\phi)\frac{d^2\varpi}{dy'^2}\right]=0,$$

or rather

(103) $$\frac{d^2\varpi}{dy'^2}=\frac{1-sin\phi}{1+sin\phi}\frac{d^2\varpi}{dx'}=tg^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\frac{d^2\varpi}{dx'^2},$$

the equation whose general integral, with two arbitrary functions f_1, f_2, is

$$(104)\quad \varpi=\frac{cos\psi}{cos(\omega+\psi)}\left\{f_1[x'-y'tg(\tfrac{\pi}{4}-\tfrac{\phi}{2})]+f_2[x'+y'tg(\tfrac{\pi}{4}-\tfrac{\phi}{2})]\right\}^*$$

* The simplicity of this result invites to find the solution of the analogous question for the elastic equilibrium, that is to find all the elastic eqilibrium modes of the mass which are near to any one of the modes studied in the preceding paragraph, and which is produced, for example, when the conditions at the walls are more complicated than those which we have admitted, or when the profil of the limiting contours is slightly curved.

Refer the medium to the coordinate axes of x' and y' parallel to the directions, everywhere the same, which affect the principal forces when one of the modes, already studied, of stable equilibrium is supposed to be realised, and call these forces at the point (x', y'), $\rho g F_1^0$, $\rho g F_2^0$ $-\rho g p_0$ their half sum, ϕ' the maximum and constant inclination, in this same mode of equilibrium, of the stresses to the prolongation of the normal to the plane elements of action, the inclination resulting, as we have seen before the formula (67^{bis}) [pp. L and LI], from the relaticn

$$sin\phi'=\frac{F_1^0-F_2^0}{2p_0}.$$

The small complementary parts $\rho g n_1$, $\rho g n_2$, $\rho g t$, which must be joined to $\rho g F_1^0$, $\rho g F_2^0$, o, to have N_1', N_2', T, will evidently satisfy equations (102) and consequently the relations (102^{bis}). The indefinite equation which will be ought to serve to the determination of ϖ will also be deduced from the formu (28^{bis}) of page XXVII (where we shall accent x, y, N_1, N_2, T) by the substitution of the following approximate values of $\frac{T'}{p}$, $\frac{N_1'-N_2'}{p}$:

$$\frac{T'}{p}=\frac{t}{p}=-\frac{1}{p_0}\,\frac{d^2\varpi}{dx'dy'},$$

$$\frac{N_1'-N_2'}{2p}\ \text{or}\ \frac{F_1^0-F_2^0+n_1-n_1}{F_1^0+F_2^0+n_1+n_2}=\frac{F_1^0-F_2^0}{F_1^0+F_2^0}\left[1+\frac{n_1-n_2}{F_1^0-F_2^0}-\frac{n_1+n_2}{F_1^0+F_2^0}\right]$$

$$=-sin\phi'+\frac{1}{2p_0}\left(\frac{d^2\varpi}{dx'^2}-\frac{d^2\varpi}{dy'^2}\right)-\frac{sin\phi'}{2p_0}\left(\frac{d^2\varpi}{dx'^2}+\frac{d^2\varpi}{dy'^2}\right).$$

To arrive at an accessible equation, it must be admitted that the function ϖ varies from point to point much more rapidly than $\frac{1}{p_0}$: which is arrived at the not very great depths l where p_0 is considerable (of the order of l), and where the derivatives of $\frac{1}{p_0}$ are of the order of $\frac{1}{p_0^2}$. Then the coefficient $\frac{1}{p_0}$, in the preceding formulae, can be supposed to be constant, andthe relation (28^{bis}), multiplied by $2p_0$, takes the homogeneous form

$$0=\left[\frac{d^4\varpi}{dx'^4}+2\frac{d^4\varpi}{dx'^2dy'^2}+\frac{d^4\varpi}{dy'^4}\right]-sin\phi'\left[\frac{d^4\varpi}{dx'^4}-\frac{d^4\varpi}{dy'^4}\right]$$

It has for general integral, with four arbitrary functions F_1, F_2, F_3, F_4,

$$\varpi=F_1\left[x'-y'tg\left(\frac{\pi}{4}-\frac{\phi'}{2}\right)\sqrt{-1}\right]+F_2\left[x'+y'tg\left(\frac{\pi}{4}-\frac{\phi'}{2}\right)\sqrt{-}\right]+F_3(x'+y'\sqrt{-1})$$
$$+F_4(x'-y'\sqrt{-1});$$

these functions can be replaced by a double infinity of terms, taken, the one, in the form $e^{my'tg\left(\frac{\pi}{4}-\frac{\phi'}{2}\right)}(Acosmx'+Bsinmx')$, the other, in the form $e^{my'}(Ccosmx'+Dsinmx')$, m, A, B, C, D denoting any constants.

It seems to me to be difficult to have some result interesting for the practice.

This expression of ϖ, differentiated twice and put in (102^{bis}), after having replaced $tg(\pi/4-\phi/2)$ by $\sqrt{(1-sin\phi)/(1+sin\phi)}$, gives

$$(105)\quad \begin{cases} n_1=\dfrac{cos\psi(1-sin\phi)}{cos(\omega+\psi)(1+sin\phi)}(f_1''+f_2''), \quad n_2=\dfrac{cos\psi}{cos(\omega+\psi)}(f_1''+f_2''), \\ t=\dfrac{cos\psi(1-sin\phi)}{cos(\omega+\psi)cos\phi}(f_1''-f_2''), \end{cases}$$

by denoting simply by f_1'', f_2'' the second derivatives of the two functions f_1, f_2, which appear in (104).

Substitute now in the formulae (100) the values (105) of n_1, n_2, t. Taking account also of the multiple proportion

$$(106)\quad \begin{cases} \dfrac{cos\psi}{cos(\omega+\psi)(1+sin\phi)}=\dfrac{sin\psi}{sin(\omega+\psi)(1-sin\phi)} \\ =\dfrac{cos\psi(1-sin\phi)}{cos(\omega+\psi)cos^2\phi}=\dfrac{sin(\omega+2\psi)}{sin(2\omega+\psi)}, \end{cases}$$

whose second and third ratios are equal to the first, the one by virtue of the equation $sin\omega-sin(\omega+2\psi)sin\phi=0$, the other identically and whose fourth is the result of the addition, term to term, of the first two after having multiplied the terms of the first by $sin(\omega+\psi)$ and those of the second by $cos(\omega+\psi)$. Thus we have simply:

$$(107)\quad \begin{cases} \dfrac{N_1'}{\rho g}=-\dfrac{sin(\omega+2\psi)}{sin2(\omega+\psi)}(1-sin\phi)(l-f_1''-f_2''), \\ \dfrac{N_2'}{\rho g}=-\dfrac{sin(\omega+2\psi)}{sin2(\omega+\psi)}(1+sin\phi)(l-f_1''-f_2''), \\ \dfrac{T'}{\rho g}=\dfrac{sin(\omega+2\psi)}{sin2(\omega+\psi)}cos\phi.\ (f_1''-f_2''.). \end{cases}$$

Such are the general formulae of the required solution. They verify exactly the two indefinite equations (101) of equilibrium and approximately the following (101^{bis}), expressing that the maximum inclination ϕ' of a stress to the prolongation of the normal to the plane element of its action attains at each point the value ϕ. In reality, the relations (107) give for the sine of the maximum inclination ϕ',

$$(108)\qquad \sin\phi' = \sqrt{\frac{4T'^2 + (N_1' - N_2')^2}{(N_1' + N_2')^2}} = \sin\phi\sqrt{1 + \frac{1}{tg^2\phi}\left(\frac{f_1'' - f_2''}{l - f_1'' - f_2''}\right)^2}.$$

The solution (107) *obtained will then be exact, if the pulverulent mass, being slightly heterogeneous, would have at each point an angle of sliding earth, ϕ', superior to ϕ by a variable quantity of the second order of infinitesimals, and given by the equation*

$$\frac{\sin\phi'}{\sin\phi} = \sqrt{1 + \frac{1}{tg^2\phi}\left(\frac{f_1'' - f_2''}{l - f_1'' - f_2''}\right)^2} = 1 + \frac{1}{2tg^2\phi}\left(\frac{f_1'' - f_2''}{l - f_1'' - f_2''}\right)^2 \text{ sensibly:}$$

we can thus admit it, in the case of a homogeneous mass, only when the ratio

$$(109)\qquad \frac{f_1'' - f_2''}{l - f_1'' - f_2''}$$

will be everywhere sufficiently small against unity, with the exception of certain very limited or quite exceptional regions, for which it will be allowable to neglect the influence on the rest.

Observe that, in the formulae (107), the function

$$f_1'', \text{ or } f_1''[x' - y'tg(\tfrac{\pi}{4} - \tfrac{\phi}{2})]$$

has the same value along each line parallel to the straight line OQ and varies only from one of these parallels to its neighboring one, so that, in the same way, the function

$$f_2'', \text{ or } f_2''[x' + y'tg(\tfrac{\pi}{4} - \tfrac{\phi}{2})]$$

has the same value along each line parallel to OQ'.

46^{bis}. *Various Formulae Relative to the Stresses.*

Now to find the expressions to which these relations (107) lead for the two components, normal $-\mathfrak{N}$ and tangential $\mathfrak{T}$, of the stress exerted on any plane element parallel to the axis of z. I will still call ε_1 the inclination of this plane element to the vertical so as to have $-(\varepsilon_1 + \psi)$ for the angle β made by its normal with the axis of x'. The general formulae (30), applied by accenting there N_1, N_2, T, substituting then for N_1', N_2', T' their values obtained from (107), after multiplying the results, in view of simplifying the second members, by $[\sin2(\omega + \phi)]/\sin(\omega + 2\psi)$, give:

$$(110)\qquad \begin{cases} \dfrac{\mathfrak{T}}{\rho g}\,\dfrac{\sin2(\omega + \psi)}{\sin(\omega + 2\psi)} = [l - f_1'' - f_2'']\sin\phi\sin2(\varepsilon_1 + \psi) \\ \qquad\qquad + (f_1'' - f_2'')\cos\phi\cos2(\varepsilon_1 + \psi), \\ \dfrac{-\mathfrak{N}}{\rho g}\,\dfrac{\sin2(\omega + \psi)}{\sin(\omega + 2\psi)} = [l - f_1'' - f_2''][1 - \sin\phi\cos2(\varepsilon_1 + \psi)] \\ \qquad\qquad + (f_1'' - f_2'')\cos\phi\sin2(\varepsilon_1 + \psi). \end{cases}$$

It will be generally preferable, instead of calculating $-\mathfrak{R}$ and $\mathfrak{T}$, to evaluate at first the inclination ϕ_1 of the stress to the prolongation of the normal to the plane element, then to deduce the resultant stress $\mathfrak{R}$ from the equality $\mathfrak{R}=\mathfrak{T}/sin\phi_1$ or

$$(111)\quad \frac{\mathfrak{R}}{\rho g}=\frac{sin(\omega+2\psi)}{sin2(\omega+\psi)}\,\frac{(l-f_1''-f_2'')sin\phi sin2(\varepsilon_1+\psi)+(f_1''-f_2'')cos\phi cos2(\varepsilon_1+\psi)}{sin\phi_1}.$$

We shall have the inclination ϕ_1 (variable at most from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$) by means of its tangent, which is equal to the ratio of the two second members of the formulae (110). By expressing that the two second members are, in fact, as $sin\phi_1$ is to $cos\phi_1$, then equating the product of the extremes to that of the means and simplifying the results, it follows

$$(112)\quad [l-f_1''-f_2''][sin\phi_1-sin\phi sin(2\varepsilon_1+2\psi+\phi_1)]$$
$$=(f_1''-f_2'')cos\phi cos(2\varepsilon_1+2\psi+\phi_1).$$

This relation permits to much simplify the formula (111). Thus find $f_1''-f_2''$ from (112) and substitute the value in (111): the expression

$$(l-f_1''-f_2'')sin\phi sin2(\varepsilon_1+\psi)+(f_1''-f_2'')cos\phi cos2(\varepsilon_1+\psi)$$

must be the product of $(l-f_1''-f_2'')/cos(2\varepsilon_1+2\psi+\phi_1)$ by

$$sin\phi[sin2(\varepsilon_1+\psi)cos(2\varepsilon_1+2\psi+\phi_1)-cos(2\varepsilon_1+\psi)sin(2\varepsilon_1+2\psi+\phi_1)]$$
$$+sin\phi_1 cos2(\varepsilon_1+\psi)=sin\phi_1[cos2(\varepsilon_1+\psi)-sin\phi],$$

and the formula (111) will be firstly reduced to

$$(113)\qquad \frac{\mathfrak{R}}{\rho g}=\frac{sin(\omega+2\psi)cos2(\varepsilon_1+\psi)-sin\phi}{sin2(\omega+\psi)cos(2\varepsilon_1+2\psi+\phi_1)}(l-f_1''-f_2'').$$

It is still simplified if we put

$$(114)\qquad \delta=\frac{\pi}{4}-\frac{\phi}{2}-\psi-\varepsilon_1,$$

that is if we call δ the inclination, to the direction OQ, of the plane element under question, the inclination being counted positively by turning from OQ toward Oy', and negatively in the contrary sense. Then the fraction

$$\frac{cos2(\varepsilon_1+\psi)-sin\phi}{cos(2\varepsilon_1+2\psi+\phi_1)}\ \text{or}\ \frac{sin(\phi+2\delta)-sin\phi}{sin(\phi-\phi_1+2\delta)}\ \text{becomes}\ \frac{2cos(\phi+\delta)\,sin\delta}{sin(\phi-\phi_1+2\delta)}.$$

Besides, after (106), the ratio $[sin(\omega+2\psi)]/sin2(\omega+\psi)$ can be replaced by $[cos\psi]/[(1+sin\phi)cos(\omega+\psi)]$, whence follows, instead of (113),

$$(115)\quad \begin{cases} \dfrac{\mathfrak{R}}{\rho g}=\dfrac{2cos\psi cos(\psi+\delta)sin\delta}{(1+sin\phi)cos(\omega+\psi)sin(\phi-\phi_1+2\delta)}(l-f_1''-f_2'') \\ \qquad =tg\left(\dfrac{\pi}{4}-\dfrac{\phi}{2}\right)\dfrac{cos\psi cos(\phi+\delta)sin\delta}{cos(\omega+\psi)}\cdot\dfrac{2(l-f_1''-f_2'')}{cos\phi sin(\phi-\phi_1+2\delta)}. \end{cases}$$

We can yet replace the factor $1/\cos(\omega+\psi)$ by the expression $[\cos(\omega-\varepsilon_1)]/[\cos(\omega+\psi)\cos(\omega-\varepsilon_1)]$ and eliminate ω in the ratio $[\cos(\omega-\varepsilon_1)]/\cos(\omega+\psi)$, as we have done for the formula (98), by means of the proportion (98^{bis}). This gives

$$\frac{\cos(\omega-\varepsilon_1)}{\cos(\omega+\psi)} \text{ or } \frac{\cos\omega\cos\varepsilon_1+\sin\omega\sin\varepsilon_1}{\cos\omega\cos\psi-\sin\omega\sin\psi}=\frac{\cos\varepsilon_1-\sin\phi\cos(2\psi+\varepsilon_1)}{(1-\sin\phi)\cos\psi};$$

we have besides, after (114),

$$2\psi+\varepsilon_1=\frac{\pi}{2}-(\phi+2\delta+\varepsilon_1),$$

and consequently

$$\begin{aligned}\cos\varepsilon_1-\sin\phi\cos(2\psi+\varepsilon_1)&=\cos[(\phi+\delta+\varepsilon_1)-(\phi+\delta)]-\sin\phi\sin[(\phi+\delta+\varepsilon_1)+\delta]\\ &=\cos(\phi+\delta)\cos(\phi+\delta+\varepsilon_1)+(\sin\phi\cos\delta+\cos\phi\sin\delta)\sin(\phi+\delta+\varepsilon_1)\\ &\qquad-\sin\phi\cos\delta\sin(\phi+\delta+\varepsilon_1)-\sin\phi\sin\delta\cos(\phi+\delta+\varepsilon_1)\\ &=\cos(\phi+\delta)\cos(\phi+\delta+\varepsilon_1)+\sin\delta\sin(\delta+\varepsilon_1).\end{aligned}$$

Thus the formula (115) is transformed to

$$(116)\quad \left\{\begin{aligned}\frac{\mathfrak{R}}{\rho g}&=\frac{\sin\delta\cos^2(\phi+\delta)\cos(\phi+\varepsilon_1+\delta)}{\cos\phi}\left[1+\frac{\sin\delta\sin(\varepsilon_1+\delta)}{\cos(\phi+\delta)\cos(\phi+\varepsilon_1+\delta)}\right]\\ &\qquad\times\frac{2(l-f_1''-f_2'')}{\cos\phi\sin(\phi-\phi_1+2\delta)\cos(\omega-\varepsilon_1)}.\end{aligned}\right.$$

The relations (115) or (116) permit to calculate $\mathfrak{R}$ at the points for which the values of the three functions l, f_1'', f_2'' will be given. But it is sufficient to know two of these functions, and also ϕ_1, that the third results therefrom and can be eliminated by the formula (112). Since $2(\varepsilon_1+\psi)=\frac{\pi}{2}-(\phi+2\delta)$, it can be written

$$[l-f_1''-f_2''][\sin\phi_1-\sin\phi\cos(\phi-\phi_1+2\delta)]=(f_1''-f_2'')\cos\phi\sin(\phi-\phi_1+2\delta),$$

or rather, under the form of proportion

$$\frac{l-f_1''-f_2''}{\cos\phi\sin(\phi-\phi_1+2\delta)}=\frac{f_1''-f_2''}{\sin\phi_1-\sin\phi\cos(\phi-\phi_1+2\delta)}.$$

Add, term to term, the numerators and the denominators, after having multiplied the two terms of the second ratio by ± 1; there result two new ratios, equal to the first,

$$\frac{l-2f_2''}{\sin\phi_1-\sin(\phi_1-2\delta)}=\frac{l-2f_2''}{2\cos(\phi_1-\delta)\sin\delta},$$

and

$$\frac{l-2f_1''}{sin(2\phi-\phi_1+2\delta)-sin\phi_1}=\frac{l-2f_1''}{2cos(\phi+\delta)sin(\phi-\phi_1+\delta)}.$$

The equation (112) is then equivalent to the continued equality

$$(117)\quad \left\{\begin{aligned} &\frac{2(l-f_1''-f_2'')}{cos\phi cos(\phi-\phi_1+2\delta)}=\frac{2(f_1''-f_2'')}{sin\phi_1-sin\phi cos(\phi-\phi_1+2\delta)}\\ &=\frac{l-2f_2''}{cos(\phi_1-\delta)sin\delta}=\frac{l-2f_1''}{cos(\phi+\delta)sin(\phi-\phi_1+\delta)},\end{aligned}\right.$$

and we see that it is sufficient to know, besides ϕ_1, any one of the quantities $l-f_1''-f_2''$, $l_1''-f_2''$, $l-2f_2''$, $l-2f_1''$, to deduce therefrom all the others. As, in each case, two at least of the three functions l, f_1'', f_2'' will be given, we shall replace, in (115) or (116), the first ratio (117) by the following ratios which will contain the two functions. Suppose for example, that these are l and f_2'': then the substitution of the third ratio (117) for the first change the two formulae (115) and (116) to this:

$$(118)\quad \left\{\begin{aligned} &\frac{\mathfrak{N}}{\rho g}=tg\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\frac{cos\psi cos(\phi+\delta)}{cos(\omega+\psi)cos(\phi_1-\delta)}(l-2f_2'')\\ &=\frac{cos^2(\phi+\delta)cos(\phi+\varepsilon_1+\delta)}{cos\phi cos(\phi_1-\delta)}\left[1+\frac{sin\delta sin(\varepsilon_1+\delta)}{cos(\phi+\delta)cos(\phi+\varepsilon_1+\delta)}\right]\frac{l-2f_2''}{cos(\omega-\varepsilon_1)}.\end{aligned}\right.$$

47. *To Take Account of the Conditions Special at the Limiting Surfaces.—Circumstances which are Shown Near the Sustaining Walls.*

All the preceding results have been deduced from the indefinite equations of equilibrium and subsist whatever the two arbitrary functions f_1'', f_2'' may be. It remains now to determine these functions so as to satisfy the conditions special to the free surface and to that which concerns to the surface of separation of the wall and the mass.

For this purpose, observe at first that the profile OA of the top slope (fig. 5, p. CII) is in the angle QOQ'. To evaluate then the three angles GOQ', GOA, GOQ. The first is equal to $y'OQ'$, or $\pi-(\frac{\pi}{4}-\frac{\phi}{2})$, decreased by ψ. The second is equal to $\frac{\pi}{2}+\omega$. Finally, the third is equal to $\frac{\pi}{4}-\frac{\phi}{2}-\psi$. We have then

$$GOQ'=(\tfrac{\pi}{2}+\omega)+\tfrac{1}{2}(\phi-\omega)+\tfrac{1}{2}(\tfrac{\pi}{2}-\omega-2\psi),\quad GOA=\tfrac{\pi}{2}+\omega,$$
$$GOQ=(\tfrac{\pi}{2}+\omega)-\tfrac{1}{2}(\phi+\omega)-\tfrac{1}{2}(\tfrac{\pi}{2}+\omega+2\psi);$$

whence result, by virtue of the two inequalities

$\omega < \phi$, $\omega + 2\psi < \frac{\pi}{2}$, (in absolute value),

that GOA is less than GOQ' and greater than GOQ, or that OA is comprised between OQ and OQ'.

This premised, the conditions at the free surface, (95), becoming, in the new system of axes

$$N_1' = 0, \ N_2' = 0, \ T' = 0 \text{ (for } l = 0, \text{ or on } OA),$$

show that, after the values (107) of N_1', N_2', T' we have

(at all the points of OA) $f_1'' + f_2'' = 0$, $f_1'' - f_2'' = 0$, or $f_1'' = 0$, $f_2'' = 0$.

Thus the function f_1'', which has the same value on all the extent of any parallel to OQ, must be zero at all the points of OA and consequently in all the space which is on the same side of OQ as the top slope OA, that is in all the part AOQ of the mass. Similarly, the derivative f_2'', invariable along each parallel to OQ, must vanish at all the points of OA and consequently in all the space which is, with respect to OQ', on the same side as the top slope OA. The sustaining wall will always be found in this space, which consequently contains the whole of the mass. *The conditions special to the free surface then oblige to make zero, in all the portion AOQ of the mass, the two functions f_1'', f_2'', or, what comes to the same thing, the small complementary parts $\rho g n_1$, $\rho g n_2$, $\rho g t$ of the forces N_1', N_2', T: they moreover oblige to suppose the function f_2'' to be zero in all the rest of the mass.*

Let us finally be occupied with the condition special to the wall. It will be either 1° that the back face of the wall falls in the angle AOQ, that is have an inclination i, to the vertical, greater than $\frac{\pi}{4} - \frac{\phi}{2} - \psi$; or 2° that the same face has, on the contrary, an inclination i less than $\frac{\pi}{4} - \frac{\phi}{2} - \psi$, and falls, with respect to OQ, on the same side as Oy'.

In the first case, the complementary terms being necessarily zero at all the points of the mass, it is impossible to satisfy the condition special to the sustaining wall however little the angle of the external friction differs from that, ϕ_1, which results from the fourth formula (96). The particular solution given by Macquorn Rankine is then thus an isolated solution, or which is not found near any other. This result tends to prove that, when a rough sustaining wall, having its back face inclined to the vertical by an angle greater than $\frac{\pi}{4} - \frac{\phi}{2} - \psi$, *begins* to overturn, the state of collapse is not produced in all the extent of the mass. Without doubt, *a corner of earth, having its base upward, makes a body with the wall and separates in block from the rest of the mass, along a plane of rupture parallel to OQ.* In fact, two contiguous couches of earth cannot slide each other, by finite quantities, without that their mutual pressure makes with the normal to their surface of separation an angle equal to that of the internal friction ϕ: the circumstance which is produced, in the extreme state of elastic equilibrium of the mass, either on either side of planes parallel to OQ or on either side of planes parallel to OQ'. Now it is natural to admit that the state of collapse, at the moment of the overturning of the wall, extends the more possibly, under the protection of the wall, the more acute is the corner of earth from those which will have some tenency to subsist: this corner must then be limited by a plane, parallel to OQ, directed upward to start from the base of the back face of the wall under consideration.

In the second case, let OM be the profile of the back face of the wall. The function f_1'', still arbitrary in the angle QOM, will be determinate, at each point of OM, such that the inclination ϕ_1 of the pressure to the prolongation of the normal to the wall is precisely equal to the given angle of the external friction. With this result, apply the general formula (112) or its transformed one (117), which determines the angle called ϕ_1 to the stresses exerted effectively on the back face of the wall: it must be there put $\varepsilon_1 = i$, and consequently

(119) $$\delta = \frac{\pi}{4} - \frac{\phi}{2} - \psi - i,$$

after having put besides $f_2'' = 0$. The second and the third member of (117) then give, by expressing f_1'' completely,

(120) $$\frac{2f_1''[x' - y'tg(\frac{\pi}{4} - \frac{\phi}{2})]}{sin\phi_1 - sin\phi cos(\phi - \phi_1 + 2\delta)} = \frac{l}{cos(\phi_1 - \delta) sin\delta} \text{ (on } OM\text{)}:$$

such is the condition special to the wall. We shall therefrom deduce the unknown function f_1'', at the various points of OM, after having put, for ϕ_1, the true value of the angle of the mutual friction of the wall and the mass.

Introduce, instead of the three variables x', y', l, the distance L of the origin O, or of the upper end of the face OM of the wall, to the point (x', y') under consideration of the same face. The normal distance, l, of the same point to the top slope, is the projection of L, taken under the angle $\omega - i$, and we shall have at first

$$l = Lcos(\omega - i).$$

Again, the straight line OM makes with the respective axes of x' and y' the angles $\frac{\pi}{2} - (\psi + i)$ and $\psi + i$: thus, on this straight line, $x' = Lsin(\psi + i)$, $y' = Lcos(\psi + i)$, and consequently

$$x' - y'tg(\frac{\pi}{4} - \frac{\phi}{2}) = \frac{Lsin(\psi + i - \frac{\pi}{4} + \frac{\phi}{2})}{cos(\frac{\pi}{4} - \frac{\phi}{2})} = -\frac{Lsin\delta}{cos(\frac{\pi}{4} - \frac{\phi}{2})}.$$

Equation (120) then becomes to be put, at the various points of OM, or when $L>0$,

$$f_1''\left[-\frac{L\sin\delta}{\cos(\frac{\pi}{4}-\frac{\phi}{2})}\right]$$

$$=-\left[\sin\phi_1-\sin\phi\cos(\phi-\phi_1+2\delta)\right]\frac{\cos(\omega-i)\cos(\frac{\pi}{4}-\frac{\phi}{2})}{2\sin^2\delta\cos(\phi_1-\delta)}\left[\frac{-L\sin\delta}{\cos(\frac{\pi}{4}-\frac{\phi}{2})}\right]$$

We see that the function f_1'', already zero for the positive values of its variable, is simply, for negative values of the latter, the product of these values by the constant factor

$$(121)\qquad A=-\frac{\sin\phi_1-\sin\phi\cos(\phi-\phi_1+2\delta)}{2\sin^2\delta}\,\frac{\cos(\frac{\pi}{4}-\frac{\phi}{2})\cos(\omega-i)}{\cos(\phi_1-\delta)}.$$

The condition special to the surface of separation of the mass and the wall thus succeeds in determining the dynamic state of the medium in the region *QOM*, by making known the value of f_1'' at each point of *OM* and consequently on all the extent of the parallel to *OQ* drawn from this point.

Observe that f_1'' vanishes at two sides *of the plane, having for profile OQ, which separates the two distinct parts of the mass in which the variations of this function are regulated by different laws.* In other words, the stresses do not cease to vary with continuity when we pass from one region to the other, although their derivatives, taken along the normals to *OQ*, are discontinuous. Such must be the case; because the equilibrium of a thin couch of matter having the plane *OQ* for one of its bases requires that the two components of the stress exerted on either side of this couch are equal each to each, the conditions which, jointed to indefinite equation (101^{bis}) and to that which expresses that the ratio of these two components under consideration is equal to $tg\phi$, show that N_1', N_2', T' have the same values on either side of *OQ*.

In practice, $\phi_1=\phi$, and the formula (121), by virtue of $1-\cos 2\delta=2\sin^2\delta$, is reduced to

$$A=-\frac{\cos(\frac{\pi}{4}-\frac{\phi}{2})\sin\phi}{\cos(\phi-\delta)}\cos(\omega-i).$$

The relation (120) gives at the same time:

$$(122)\quad (\text{on } OM)\; f_1''=\frac{\sin\phi\sin\delta}{\cos(\phi-\delta)}l,\; l-f_1''=\frac{\cos\phi\cos\delta}{\cos(\phi-\delta)}l,\; \frac{f_1''}{l-f_1''}=tg\phi\, tg\delta.$$

At the various points of any parallel to *OQ*, taken in the mass to start from a point of OM, the constant quantity f_1'' is then positive, the difference $l-f_1''$ being the greater the remoter is it along this parallel. The ratio $f_1''/(l-f_1'')$ in consequence attains its greatest value at the departure, where it is equal to $tg\phi tg\delta$, and the formula (108) [in which we have $f_2''=0$], permits to be put

$$(122^{bis}) \qquad \frac{sin\phi'}{sin\phi}\begin{cases} >1, \\ <\sqrt{1+tg^2\delta}=\dfrac{1}{cos\delta}. \end{cases}$$

Whenever the positive inclination, δ, of the back face of the wall to the direction OQ, will not be very great (or will be less, for example, than the angle $\frac{\pi}{8}=22°\frac{1}{2}$, whose secant $1/cos\delta$ is only 1,082), we can suppose, save negligible evrors, $\phi'=\phi$, and regard the approximate solution to be applicable.

Let us now be occupied with the stress undergone by the unit area of the face OM, at the distance $L=l/cos(\omega-i)$ from its top end. The relation (118), where the inclination ϕ_1 of the pressure to the prolongation of the normal to OM will be equal to the known angle of the mutual friction of the wall and the mass, gives it immediately, when we make therein $\varepsilon_1=i$, $\delta=\frac{\pi}{4}-\frac{\phi}{2}-\psi-i$, and besides $f_2''=0$. Consequently, the laws of the pressure will be those which we have found at Art. 44 (p. XCIX), except that ϕ_1 will be different and the numerical coefficient $K=\mathfrak{R}/(\rho gL)$ will have the value

$$(123) \qquad \begin{cases} K=tg\left(\dfrac{\pi}{4}-\dfrac{\phi}{2}\right)\dfrac{cos\psi cos(\phi+\delta)cos(\omega-i)}{cos(\phi_1-\delta)cos(\omega+\psi)} \\ \quad =\dfrac{cos^2(\phi+\delta)cos(\phi+i+\delta)}{cos\phi cos(\phi_1-\delta)}\left[1+\dfrac{sin\delta sin(i+\delta)}{cos(\phi+\delta)cos(\phi+i+\delta)}\right]. \end{cases}$$

This formula can take place: 1° of the more general one (fifth 96) which Mr. Maurice Levy has obtained and which holds when ϕ_1 has precisely the value resulting from the fourth relation (96); 2° of the very simple formula (99) given by Mr. Maurice Levy for the case in which $\phi_1=\phi$ and $\delta=0$: in fact, the last member of (123) reduces then to $cos(\phi+i)$. But it extends furthur to all the limiting equilibrium modes in tension which are sufficiently near to it, or for which, δ being also positive, the angle ϕ_1 of the external friction is not much different from that which verifies the fourth relation (96).

Apply it to the most simple particular case which is that of a horizontal earth surface sustained by a vertical wall. Then we have

$$\omega=0,\ \psi=0,\ i=0,\ \delta=\tfrac{\pi}{4}-\tfrac{\phi}{2},$$

and consequently

$$(123^{bis})\ K=tg\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\frac{cos(\frac{\pi}{4}+\frac{\phi}{2})}{cos[\phi_1-(\frac{\pi}{4}-\frac{\phi}{2})]}=tg^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\frac{cos(\frac{\pi}{4}-\frac{\phi}{2})}{cos[\phi_1-(\frac{\pi}{4}-\frac{\phi}{2})]}.$$

The coefficient K is the same and is equal to $tg^2(\frac{\pi}{4}-\frac{\phi}{2})$, either when the wall is supposed to be infinitely smooth or $\phi_1=0$, the hypothesis in accordance with the fourth formula (96) for zero ω and i, or when ϕ_1 is complement of ϕ or

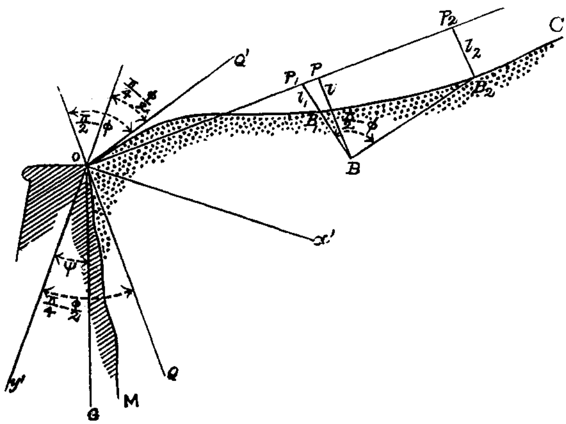
$$\phi+\phi_1=\tfrac{\pi}{2},$$

as we arrive when we take $\phi=\phi_1=45^\circ$. The values of ϕ_1 intermediate between 0 and $\frac{\pi}{2}-\phi$ give the value of K a little smaller: the least of them correspond to $\phi_1=\frac{\pi}{4}-\frac{\phi}{2}$; it is less than $tg^2(\frac{\pi}{4}-\frac{\phi}{2})$ in the ratio of 1 to $cos(\frac{\pi}{4}-\frac{\phi}{2})$, or about of 1 to $cos 22^\circ\frac{1}{2}=0{\cdot}9239$ if $\phi=45^\circ$.

48. *Study of the Case in which the Profiles of the Top Slope and the Back Face of the Wall are Curved.*

The results established in Arts. 46 and 46^{bis} can be extended to a mass whose top profile OB_1B_2C (fig. 6) will be slightly curved and which will be sustained by a wall having its back face OM equally curved. I will suppose that we have taken the origin of the coordinates x', y' at the intersec-

Fig. 6.



tion O of this face and the top slope; also, I will admit that the variable inclination of this same slope to a certain straight line OA remains small, or that the declivity of the top surface of the mass does not deviate much from the constant ω measuring the declivity of OA. Lastly, for the sake of simplicity, I will suppose that the profile of the back face OM of the wall is wholly on one side of the straight line OQ, and also that it deviates continuously from this straight line, starting from the point O, so as to be cut at one point only by each parallel to OQ.

Consider any point $B(x', y')$ of the mass and take, starting from this point, parallel to the two fixed directions QO, OQ', the two straight lines BB_1, BB_2, to the limit of the mass. If the point B is in the region QOC, the extremities B_1, B_2 are evidently found on the free surface OC, and we can take, from the three points B, B_1, B_2, on the straight line OA produced indefinitely in both directions, the perpendiculars BP, B_1P_1, B_2P_2, which I will call l, l_1, l_2 respectively: these perpendiculars will be evaluable for each position of the point B, either graphically or analytically as functions of x', y', when we shall have given the top profile OC and consequently a straight line OA inclined slightly to this profile; I will also compute them positively down from OA and negatively up. If, on the contrary, the point B is in the region comprised between OQ and the prolongation of $Q'O$, which can happen in the case only when the wall OM is outside of the angle QOQ', B_2 will always be situated on the free surface OC, but B_1 will be found on the profile OM of the wall: then there will be on OA the two perpendiculars l, l_2 only. Observe that, in all the cases, the two differences $l-l_1$, $l-l_2$ are positive, because, OA being in the angle QOQ', the points B_1, B_2 will be found above a parallel through B to OA.

Observe now how the indeterminateness of the two arbitrary functions f_1'', f_2'' will permit to adapt to the actual problem the integrals (107) of the indefinite equations. Try at first to satisfy the special conditions at the free surface, by expressing that N_1', N_2', T' vanish when the point B belongs to the curve OC. After the formulae (107), it is necessary and sufficient, for this, that we have:

$$(\text{on } OC), \quad l-f_1''-f_2''=0, \quad f_1''-f_2''=0, \quad \text{or} \quad f_1''=f_2''=\tfrac{1}{2}l.$$

The two functions f_1'', f_2'' are thus determined on the top slope and, as they have the same values along any parallel to OQ or OQ', they are determinate, by the very fact, the first in all the region QOC, and second in all the extent of the mass. At any point B, we have then

$$(124) \qquad f_2''=\tfrac{1}{2}l_2,$$

and we have also, but only when this point belongs to the region QOC,

(125) $f_1'' = \frac{1}{2} l_1$ (in the region QOC).

The solution obtained being admissible, for a homogeneous mass, only when the ratio (109) [p. CXI] is a small quantity, it must be examined whether the expressions (125) and (124) of f_1'', f_2'' give small values to this ratio thus becoming

$$(125^{bis}) \qquad \frac{\frac{1}{2}(l_1 - l_2)}{l - \frac{1}{2}(l_1 + l_2)}.$$

Now, the profile B_1 B_2 C being supposed to be slightly inclined to OA, the denominator of (125^{bis}) is always comparable, or even sensibly equal, to the distance of the point B under consideration to the top slope, while the numerator of (125^{bis}) is of the order of the product of the small inclination of B_1 B_2 to OA by the straight line P_1 P_2, which is comparable to the distance of the point B to the top slope. Hence the solution is acceptable so long as the inclinations, to the horizon, of the various parts of the top slope do not differ much from a constant ω.

In summary, the equilibrium mode, in the principal region QOC, is completely determined by the indefinite equations and the conditions special to the free surface. Consequently, if the back face OM. of the sustaining wall is comprised in this region, we can satisfy the condition relative to it only when the stress exerted on each of its plane elements, after the formulae (110), makes precisely, with the prolongation of the corresponding normal, an angle ϕ_1 equal to that of the given external friction. This shows that in general a corner of earth adjacent to the wall cannot collapse, and that there will be rupture, with finite sliding, between this corner of earth and the rest of the mass. The surface of rupture being bound to the geometrical position of plane elements on which the angle of the stress and the normal attains the maximum value ϕ (or more exactly ϕ'), its profile will make up successively, following one another and starting from the base of the face OM, a series of infinitely small straight lines whose inclination ε_1 relative to the vertical will be the such, at each point, that the formulae (110) give $\mathfrak{T}/(-\mathfrak{N}) = tg\phi'$. The direction of the tangents to the curve thus obtained will generally not deviate much from that of OQ. In the calculation of the dimensions of a wall capable to sustain, without overturning, the limiting pressure which it undergoes at the moment when the state of the mass becomes collapsing, we can regard the corner of earth adjacent to the wall to make up a body with the latter itself by maintaining it at the elastic state by its frictions, or can reason as if the surface of rupture is the true back face of this wall.

Pass on now to the case in which the mass comprises, besides the region *QOC*, another small region *QOM*, in which the function f_1'', constant along each parallel to *OQ*, will be disposable. We can determine f_1'' so that the inclination ϕ_1 of the stress exerted on the various plane elements of the face *OM*, relative to the normal to these plane elements, is precisely equal to the given angle of the external friction. For this purpose, if ϕ_1 denote this angle, i the inclination, at the point under consideration, of *OM* to the vertical, and consequently δ its inclination, $\frac{\pi}{4}-\frac{\phi}{2}-\psi-i$, to *OQ*, it is sufficient, after the second and the third memder of (117), to have

$$(126) \qquad f_1''-f_2''=\frac{\sin\phi_1-\sin\phi\cos(\phi=\phi_1+2\delta)}{2\cos(\phi_1-\delta)\sin\delta}(l-2f_2'') \quad \text{(on } OM\text{)}.$$

Now, after (124), $2f_2''=l_2$, and besides $\phi_1=\phi$ in practice, which reduce the expression $\sin\phi_1-\sin\phi\cos(\phi-\phi_1+2\delta)$ to $2\sin\phi\sin^2\delta$. The preceding formula then becomes

$$(127) \qquad f_1''=\tfrac{1}{2}l_2+\frac{\sin\phi\sin\delta}{\cos(\phi-\delta)}(l-l_2) \quad \text{(on } OM\text{)}.$$

Knowing f_1'' at the various points B_1 of *OM*, we shall have this function at the corresponding points *B* of the region *QOM*, whose mechanical state, regulated by other laws than that in the region *QOC*, will thus be completely determined. We can yet prolong on this side of the point *O* the straight line *OA* and also, *fictitiously*, the profile of the top slope, by maintaining, between the latter prolongation on one hand, to the point it will meet any parallel to *OQ*, and, on the other, the prolongation of *OA*, a distance l_1 double of the value (127) which the function f_1'' acquires at the intersection of this parallel and *OM*. It is clear that the ficticious part thus added to the mass before *OM* will produce, on the real mass *MOC*, the same effect as the sustaining wall; the problem of the limiting equilibrium of a mass limited by a wall then is reduced to that of the limiting equilibrium of a laterally indefinite mass. Observe that its top profile will present in general, at the origin *O*, an angular point, and that in consequence the derivative of f_1'', or those of the forces N_1', N_2', T' in a direction normal to *OQ*, will take the different values on the two sides of this straight line, although the function f_1'' and these forces themselves remain continuous.

Observing that $f_2''=\frac{1}{2}l_2$, the relation (127) gives

$$(128)\quad \begin{cases} (\text{on } OM),\, f_1''-f_2''=\dfrac{sin\phi\, sin\delta}{cos(\phi-\delta)}(l-l_2), \\ l-f_1''-f_2''=\dfrac{cos\phi\, cos\delta}{cos(\phi-\delta)}(l-l_2), \quad \dfrac{f_1''-f_2''}{l-f_1''-f_2''}=tg\phi\, tg\delta. \end{cases}$$

After these formulae, the two expressions $f_1''-f_2''$ and $l-f_1''-f_2''$ are positive, as $l-l_2$, at the various points of OM. Along a parallel to OQ, in the real mass, starting from a point of OM, the increasing quantity l varies besides much more rapidly than the function $f_2''=\frac{1}{2}l_2$, whose increments are proportional to the product of quantities comparable to those of l, multiplied by the small inclination of the curve B_2C to OA: hence, the greatest value which the ratio $(f_1''-f_2'')/(l-f_1''-f_2'')$ receives at the same time must be in general its initial value $tg\phi tg\delta$ When it is thus, the formula (108) gives, as in the preceding Article (form. 122^{bis}),

$$(129)\qquad \frac{sin\phi'}{sin\phi}\begin{cases} >1 \\ <\dfrac{1}{cos\delta}; \end{cases}$$

which signifies that the solution as found can be admitted whenever the positive inclination δ, to the direction OQ, of the various parts of the back face of the wall will have its cosine sufficiently near to unity, or will not exceed, for example, $\pi/8=22°\frac{1}{2}$. The conclusion will be the same if there are the points at which the mumerator $f_1''-f_2''$ increases, along a parallel to OQ, in a proportion as rapidly as the denominator $l-f_1''-f_2''$: the ratio $(sin\phi')/sin\phi$ does not then cease to be slightly greater than unity, because the fraction $(f_1''-f_2'')/(l-f_1''-f_2'')$ will continue to be comparable to $tg\phi tg\delta$.

It remains us to evaluate the stress exerted by the mass on the wall. We know already the direction of that which is applied to any plane element of the face OM, since it makes, with the prolongation of the normal to the plane element under consideration, the known angle ϕ_1 of the external friction. Besides its magnitude $\mathfrak{R}$, per unit of surface, will be given by the formula (118), where it will be sufficient for the purpose to replace $2f_2''$ by l_2, ε_1 by the inclination i, to the vertical, of the same plane element, and consequently δ by $\frac{\pi}{4}-\frac{\phi}{2}-\psi-i$. All these elementary pressures, being comprised in the same plane, will be equivalent to a unique force; but its calculation will be generally complicated.

The result becomes somewhat simple in the case only when the back face of the wall is plane and of the same degree of smoothness or of roughness in all its extent, the bypotheses in virtue of which all the elementary stresses are parallel and of the same sense. Call then dL the magnitude of a band of this face; $L=l/cos(\omega-i)$ the distance of the points of this band to the intersection of the face under consideration and of the top slope; $L_2=l_2/cos(\omega-i)$, the analogous distance, measured parallel to the back face of the wall, of each point as B_2 to the straight line OA, the distance which will become a known function of L since all the points B will be taken on OM. The elementary pressure $\mathfrak{R}dL$, undergone by the unit length of the band, will be equal to

$$\mathfrak{R}dL=\rho g tg\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\frac{cos\psi cos(\phi+\delta)cos(\omega-i)}{cos(\phi_1-\delta)cos(\omega+\psi)}(L-L_2)dL$$

$$=\rho g\frac{cos^2(\phi+\delta)cos(\phi+i+\delta)}{cos\phi cos(\phi_1-\delta)}\left[1+\frac{sin\delta sin(i+\delta)}{cos(\phi+\delta)cos(\phi+i+\delta)}\right](L-L_2)dL.$$

The total pressure per unit length of the wall, up to a distance $\mathfrak{L}$ of the top end, will then have for its expression $\int_0^{\mathfrak{L}}\mathfrak{R}dL$, or

$$(130)\qquad P=K\frac{\rho g\mathfrak{L}^2}{2}\int_0^{\mathfrak{L}}\left(1-\frac{L_2}{L}\right)\frac{2LdL}{\mathfrak{L}^2},$$

if we call always K the constant coefficient defined by the formula (123).

As to the distance, $\mathfrak{L}_1$ of the point of application of this pressure to the top end, it will result from the equation of moments

$$\mathfrak{L}_1P=\int_0^{\mathfrak{L}}\mathfrak{R}LdL=K\frac{\rho g\mathfrak{L}^3}{3}\int_0^{\mathfrak{L}}\left(1-\frac{L_2}{L}\right)\frac{3L^2dL}{\mathfrak{L}^3},$$

and will cousequently be equal to

$$(131)\qquad \mathfrak{L}_1=\frac{2}{3}\mathfrak{L}\frac{\int_0^{\mathfrak{L}}\left(1-\frac{L_2}{L}\right)\frac{3L^2dL}{\mathfrak{L}^3}}{\int_0^{\mathfrak{L}}\left(1-\frac{L_2}{L}\right)\frac{2LdL}{\mathfrak{L}^2}}.$$

The determination of the pressure P and of its point of application will thus necessitate the calculation of two integrals, whose numerical evaluation will be effected when L_2 will be given as a function of L, that is when the exact profile of the top slope will be known.

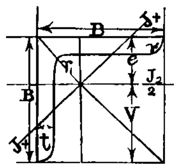
For $L_2=0$, we find the results already obtained in the preceding Article.

The case in which the top surface OC is plane and the sustaining wall

OM is curved seems to be the most simple after that which we have examined. We can put then, in the general formula (118), $f_2''=0$; but the angles i (or ε_1) and δ can no more be supposed to be constant, the circumstance which must render the calculations less accessible.

日本製鐵所ニテ現今製出シ得ル山形軟鋼簡易使用表 (I)

浦上正二郎君寄稿



$t_{min.}=0.1B$; B<100ノ時

$t_{max.}=\frac{1}{11}B$; B>100ノ時

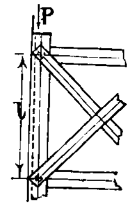
G＝壹メーターニ付キ重量瓩 但シ壹立粍ニ付 7.85トシテ計算ス

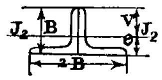
$r_1=\frac{t_{min.}+t_{max.}}{2}$

$r=0.5r_1$

P＝縮壓或ハ伸張強度 瓲ニテ

l＝實用荷重Pニ對スル最大長＝$\sqrt{\frac{J_x}{2.5P}}$



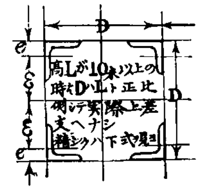


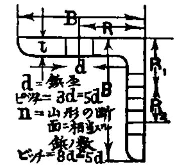
$e \backsimeq {}^1/_4B+0.366t$

J ＝ 貳個山形ノ力率

W_2＝横曲力率＝$\frac{J_2}{V}$

横折強(瓩米分)＝$W_2 750$





D＝荷重P_1ニ對シ高L10米ノ時柱ノ幅

$n=\frac{F}{\frac{\pi}{4}d_2}$

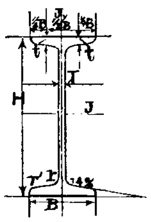
$\varepsilon=\sqrt{\frac{2.5P_1L^2-2J_2}{4.F.}}$; P_1＝最大實用荷重＝4.F.750.

B (m.m.)	r	r_1	t (粍)	F (□c.m.)	G (瓩)	P＝F·750 (瓲)	J_x (cm.4)	$l_{max.}$ (m.m.)	e (m.m.)	J_2 (c.m.4)	W_2 (c.m.3)	4F (□c.m.)	P_1＝4F.750 (瓲)	D (m.m.)	d (吋)	n	R (m.m.)	R_1 (m.m.)	R_2 (m.m.)	B (粍)
15	2	35	3	0.82	0.64	0.6	.06	200	4.8	0 3	0.29	3.28	2.45	280	3/16″	5	9			15
			4	1.05	0.86	0.8	.08	200	5.1	0.37	0.37	4.2	3.15	285						
20	2	35	3	1.12	0.81	0.9	.15	260	6.	0.77	0.55	4.48	3.35	300	1/4″	4	12			20
			4	1.45	1.13	1.1	.19	290	6.4	0.06	0.71	5.8	4.35	260						
25	2	35	3	1.42	1.11	1.1	.31	335	7.3	1.56	0.89	5.68	4.25	290	1/4″	6	15			25
			4	1.85	1.44	1.4	.40	340	7.6	2.01	1.15	7.4	5.55	285						
30	2.5	5	4	2.27	1.77	1.8	.76	410	8.9	3.61	1.71	9.07	6.7	290	3/8″	4	18			30
			6	3.27	2.55	2.6	1.06	405	9.6	4.96	2.43	13.1	9.8	290						
35	2.5	5	4	2.07	2.08	2.16	1.24	485	10.	5.92	2.37	10.7	8.0	290	3/8″	5	20			35
			6	3.87	3.02	3.04	1.77	480	10.8	8.26	3.41	15.5	12.6	295						
40	3	6	4	3.08	2.40	2.4	1.86	555	11.2	8.94	3.11	12.3	9.2	298	1/2″	4	23			40
			6	4.48	3.49	3.5	2.67	550	12.0	12.7	4.52	17.9	13.4	295						
			8	5.80	4.52	4.5	3.38	545	12.8	15.8	5.80	23.2	17.4	293						
45	3.5	7	5	4.3	3.30	3.4	3.25	618	12.8	15.7	4.87	17.2	12.8	290	1/2″	5	25			45
			7	5.86	4.57	4.6	4.39	616	13.6	20.8	6.63	23.5	17.5	295						
			9	7.34	5.73	5.7	5.40	615	14.4	25.2	8.25	29.4	22.0	300						
50	3.5	7	5	4.8	3.75	3 8	4.59	695	14.0	22.0	6.10	19.2	14.2	300	5/8″	4	28			50
			7	6.56	5.12	5.1	6.02	690	14.9	29.1	8.30	26.3	19.6	312						
			9	8.24	6.43	6.4	7.67	685	15.6	35.8	10.39	33.0	27.6	315						
55	4	8	0	6.31	4.92	4.9	7.24	770	15.6	34.0	8.79	25.2	19.0	300	5/8″	5	30			55
			8	8.23	6.42	6.4	9.35	765	16.4	44.2	11.5	32.8	24.6	302						
			10	10.07	7.85	7.9	11.27	760	17.2	52.7	13.9	40.3	30.4	305						
60	4	8	6	6.09	5.39	5.4	9.43	835	16.9	45.5	10.6	27.6	20.8	300	5/8″	5	34			60
			8	9.03	7.04	7.6	12.1	800	17.7	5.5.3	13.8	36.1	27.2	302						
			10	11.07	8.63	8.6	14.6	825	18.5	69.7	16.8	44.3	32.0	305						
65	4.5	9	7	8.7	6.79	6.8	13.8	900	18.5	66.8	14.4	34.8	25.6	300	11/16″	5	36			65
			9	11.0	8.50	8.6	17.2	895	19.3	82.6	18.1	43.9	32.0	305						
			11	13.2	10.30	10.3	20.7	898	20.0	97.5	21.7	52.7	36.4	305						
70	4.5	9	7	9.4	7.33	7.3	17.0	980	19.7	84.0	16.8	37.6	27.5	300	3/4″	4	38	26	22	70
			9	11.9	9.20	9.3	22.0	975	20.5	105.	21.2	47.5	35.2	305						
			11	14.3	11.13	11.1	26.0	965	21.3	124.	25.4	57.1	41.6	310						
75	5	10	8	11.5	8.94	8.9	24.4	1048	21.3	118.	21.9	45.9	32.0	295	3/4″	5	42	28	25	75
			10	14.1	11.00	11.0	29.8	1040	22.1	142.	26.9	56.4	41.6	300						
			12	16.7	13.00	13.0	34.7	1035	22.9	165.	31.7	66.7	48.0	305						
80	5	10	8	12.3	9.57	9.6	29.0	1110	22.6	144.	25.1	49.1	35.2	300	3/4″	6	45	30	27	80
			10	15.1	11.78	11.8	35.9	1100	23.4	175.	30.9	60.4	44.8	302						
			12	17.9	13.94	13.9	43.0	1120	24.1	204.	36.4	71.5	52.0	305						
100	6	12	10	19.2	14.90	14.9	73.3	1415	28.2	354.	49.3	76.6	57.5	310	7/8″	6	55	35	36	100
			12	22.7	17.7	17.7	86.2	1400	29.0	414.	58.3	90.9	68.0	315						
			14	26.2	20.4	20.4	98.3	1385	29.8	470.	67.0	104.8	78.0	320						
120	6.5	13	11	25.4	19.8	19.8	140.0	1680	33.6	680.	78.8	101.	75.5	325	7/8″	7	66	40	45	120
			13	29.7	23.2	23.2	162.0	1670	34.4	787.	92.1	118.	88.0	330						
			15	33.9	20.5	26.5	186.0	1660	35.1	891.	105.	136.	102.	335						
140	7.5	15	13	35.0	27.3	27.3	262.0	1900	39.2	1276.	127.	140.	105.	335	1″	8	78	46	55	140
			15	40.0	31.2	31.2	298.0	1950	40.0	1446.	145.	160.	120.	340						
			17	45.0	35.1	35.1	334.0	1940	40.8	1610.	162.	180.	135.	345						
160	8.5	17	15	46.1	35.9	35.9	453.0	2250	45.0	2198.	191.	184.	138.	350	1″	10	90	52	65	160
			17	51.8	40.4	40.4	506.0	2240	46.0	2451.	214.	207.	155.	350						
			19	57.5	44.9	44.9	558.0	2230	46.0	2695.	237.	230.	172.	350						

By. S. Urakami. 1904.

日本製鐵所ニテ現今製出シ得ル工字及溝形軟鋼簡易使用表 (II)

浦上正二郎君寄稿

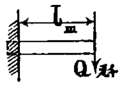


$B = 0{\cdot}4H + 10$ $T = .03H + 1.5$ …………………… H250以下ノ片

$B = 0.3H + 35$ $T = .036H$ …………………… H250ヨリ大ナル片

$r = T$ $r' = 0.6T$ t ハBノ$\frac{1}{4}$ノ所ニテ測ル

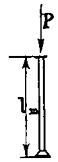
F = 断面平方積□c.m.   G = 壹米ノ重量瓩



Ql = 横曲力 (米瓩ニテ)

W = 横曲力率 $= \frac{J}{\frac{1}{2}H}$

$Ql = \frac{750W}{100} \gtreqless S$

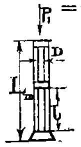


P = 最大荷重トン

$P = .75F$

$J_y = 2.5Pl^2$

$l = \sqrt{\frac{Jy}{2.5P}}$

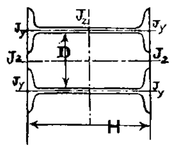


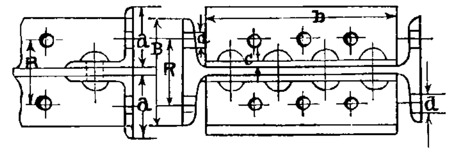
P_1 = 最大荷重 = $2P$ 瓩ニテ

$P_1 = 2F.700$

$J_2 = 2{\cdot}5P_1L^2 = 2J$

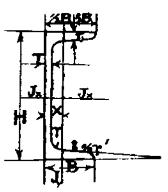
$L = \sqrt{\frac{J_x}{2.5P}}$





$D = 2\sqrt{\frac{J_2 - J_y}{2F}}$ $J_2 = 2\left(Jy + \frac{D^2}{4}F\right)$ $d = T + 8.5$ $a = .15H + 20$ $b = 85H - 10$

H (m.m.)	B (m.m.)	T (m.m.)	t (m.m.)	r (m.m.)	r^1 (m.m.)	F (□c.m.)	G (kg/m.)	$P = .75F$ (ton.)	Jy (c.m.4)	l (m.)	J (cm4)	$W = \frac{J}{\frac{1}{2}H}$ (cm3)	$S = 750W$ (m.kg)	$P_1 = 2F.75$ (ton)	J_2 (c.m.4)	D (m.m.)	L (m.)	d (inch)	R (m.m.)	c (m.m.)	b (m.m.)	a (m.m.)	H (m.m.)
100	50	4.5	6.8	4.5	2.7	10.6	8.28	7.4	12.2	.810	170	34.1	255	14.8	340	78.6	3.0	1/2″	30	4	75	35	100
120	58	5.1	7.7	5.1	3.1	14.2	11.1	9.95	21.4	.925	327	54.5	410	19.9	654	94.5	3.6	9/16″	34	4	92	40	120
150	70	6.0	9.0	6.0	3.6	20.4	15.9	14.3	43.7	1.11	734	97.9	735	28.6	1468	118.	4.5	9/16″	41	5	118	45	150
180	82	6.9	10.4	6.9	4.1	27.9	21.7	19.6	81.3	1.29	1444	161	1210	39.2	2888	145.	5.4	5/8″	47	5	142	45	180
200	90	7.5	11.3	7.5	4.5	33.4	26.1	23.5	117.	1.41	2139	214	1600	47.0	4278	158	6.0	5/8″	51	5	158	50	200
220	98	8.1	12.2	8.1	4.9	39.5	30.8	27.6	163.	1.54	3055	278	2080	55.2	6110	173	6.6	5/8″	56	6	175	55	220
240	106	8.7	13.1	8.7	5.2	46.1	35.9	32.2	220.	1.65	4239	353	2650	64.4	8478	189	7.2	3/4″	60	6	192	55	240
260	113	9.4	14.1	9.4	5.6	53.3	41.6	37.5	287.	1.75	5735	441	3300	75.0	11470	205	7.8	3/4″	65	6	208	60	260
280	119	10.1	15.2	10.1	6.1	61.0	47 6	42.5	363	1.85	7575	541	4050	85.	15150	219	8.4	3/4″	69	8	225	60	280
300	125	10.8	16.2	10.8	6.5	69.0	53.8	48.5	449	1.92	9758	652	4900	97.	19516	235	8.9	7/8″	73	8	240	65	300
400	155	14.4	21.6	14.4	8.6	118.0	91.8	82.5	1160	2.37	29173	1459	10950	165.	58346	312	11.9	1″	95	9	320	80	400

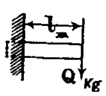


$B = .25H + 25$ } * NP.ニ於テ

$r = t$ $r' = \frac{1}{2}t$

F = 断面平方積□センチ

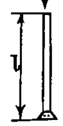
G = 壹米ノ重量瓩ニテ



W = 横曲力率 $= \frac{J_x}{\frac{1}{2}H}$

$S = \frac{750W}{100.}$ 米瓩ニテ

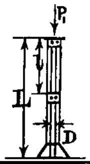
lQ = 横曲力 $\gtreqless S$



P = 壹本ニ對スル最大荷重トン = .7F

$J_y = 2.5Pl^2$

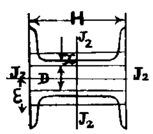
l = P ニ對スル最大長 $= \sqrt{\frac{J_y}{2.5P}}$

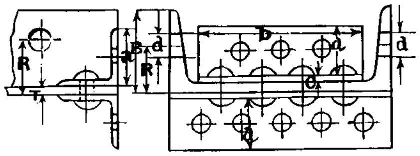


$P_1 = 2P = 2F$ ニ對スル最大荷重トン

$J_2 = 2.5P_1L^2$

$L = P_1$ニ對スルJ_2ノ最大長 $= \sqrt{\frac{J_2}{2.5P}}$ $\varepsilon = \sqrt{\frac{J_2 - J_y}{2F}}$ $J_2 = 2(J_y + \varepsilon^2 F)$





H (粍)	B (粍)	T (粍)	t (粍)	r (粍)	r_1 (粍)	F (□糎)	G (瓩)	$P = .7F$ (噸)	Jy (糎4)	l (米)	J_x (糎4)	W (糎3)	$S = 7.50W$ (米瓩)	x (粍)	$P_1 = 2P$ (噸)	J_2 (糎4)	D (粍)	L (米)	d (吋)	R (粍)	c (粍)	b (粍)	a (粍)	H (粍)
105	65	8	8	8	4	17.3	13.5	12.0	61.2	1.4	287	54.7	400	18.8	24	574	34.6	3.1	5/8″	38	4	70	40	105
117.5	65	10	10	10	5.	22.6	17.6	15.8	77.1	1.4	447	76.1	570	19.1	32	894	42.7	3.3	3/4″	40	5	75	45	117.5
145	60	8	8	8	4	19.8	15.4	14.0	53.6	1.2	585	80.7	600	15.0	28	1170	73.6	4.1	5/8″	35	4	110	40	145
180 *	70	8	11	11	5.5	28.0	21.8	19.5	114	1.5	1354	15.0	1120	19.2	39	2708	94.7	5.2	3/4″	42	5	134	45	180
235	90	10	12	12	6	42.4	33.1	29.5	272	1.9	3429	292	2180	22.8	59	6858	127	6.8	7/8″	52	6	183	55	235
260 *	90	10	14	14	7	48.3	37.7	33.5	317	1.9	4823	371	2780	23.6	67	9646	146	7.5	7/8″	54	6	200	60	260
300 *	100	10	16	16	8	58.8	45.8	41.5	495	2.1	8026	535	4000	27.0	83	16052	172	8.7	1″	60	7	235	70	300

By S. Urakami, 1904.