§ VIII.

SOLUTION OF THE PROBLEMS OF EQUILIBRIUM OF GREATEST IMPORTANCE IN APPLICATIONS BY MEANS OF A CONDITION OF STABILITY WHICH SATISFIES THE SPECIAL RELATIONS AT THE WALLS.

37. *Case in which the Most Stable Equilibrium is Realised.*

The formulae established in the preceding Article resting on the special conditions (37) [p. XXXI] hold true only for the case of a mass initially free from any stress and devoid of weight, which has assumed a new equilibrium in becoming heavy, without that the couch contiguous to the back face of the wall which sustains it shall be displaced when the wall is rough or deviated from its primitive plane when it is smooth. Now we have seen at the end of § III that the immobility of the particles adjacent to a rough wall for example rather leads to regard their displacements u, v as functions of x, y, determinate in each case, but not equal to zero, and that in consequence the simple conditions $u=0$, $v=0$ can scarcely be verified in practice.

In fact, we form a pulverulent mass by depositing successively against a rough wall made at the outset of the couches of earth or of sand. The particles near the wall are almost undisturbed in the positions of natural state so long as they are not too much compressed; but the increasing and quite considerable pressure which they have afterwards to sustain causes a large number of finite slidings or of partial collapsings, in consequence of which an entirely different mode of equilibrium is realised. This mode must be the most possibly favorable to the internal stability of the mass, that is that for which the maximum dilatation ∂_1 acquires at the various points its least values compatible with the degree of resistance which the wall can oppose: because the equilibrium mode thus defined, if it had not already been completely realised at an instant after we have deposited the last couches of earth or of sand, would not be delayed to be established by the effect of the small shocks due to many causes and undergone by the mass at almost every instant, and which permit the sandy grains to be grouped in the least constrained manner.*

Considering a zone contiguous to the retaining wall and in the extent of which the immediate influence of the latter produces perhaps the local perturbations, all the equilibrium modes realizable to the mass are represented by the formulae (57) to (60) [p. XXXVII and XXXVIII], in which ε denotes a parameter varying from $\frac{\omega-\tau}{2}$ to $\frac{\omega+\tau}{2}$ (form. 70^{bis}).** Now, after (60), ∂_1 is the least possible when $\varepsilon=\frac{\omega}{2}$.

Such is the value of the parameter ε *which, by reducing the maximum dilatation* ∂_1 *to its minimum* $\pm(\sin\omega)/2m$, *corresponds to the most internally stable mode of equilibrium or one the possibly near to the natural state for which we shall have everywhere* $\partial_1=0$, *and consequently to the equilibrium mode which will be produced if the retaining wall is firm enough to sustain the corresponding pressure without overturning.*

Suppose then $\varepsilon=\omega/2$, and, calling ε_1 the inclination of the back face of the wall to the vertical, calculate the pressure at first. To obtain it, it is sufficient to make $\varepsilon=\omega/2$ in the formulae (82); it follows then

$$(83)\quad tg(\phi_1+\varepsilon_1+\tfrac{\pi}{4}-\tfrac{\omega}{2})=\frac{tg(\varepsilon_1+\frac{\pi}{4}-\frac{\omega}{2})}{tg^2(\frac{\pi}{4}-\frac{\omega}{2})},\quad K=\frac{tg\omega\cos(\omega-\varepsilon_1)\cos(\omega-2\varepsilon_1)}{\sin\phi_1}.$$

The pressure will then be governed by the laws expressed in the formulae (82), except that the angle ϕ_1 and the coefficient K shall have the completely determinate values resulting from (83)

The expression of K becomes extremely simple when the back face of the wall is vertical or when we have $\varepsilon_1=0$ and consequently $\phi_1=\omega$: it is reduced to

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* The vibratory movements produce in the long run an analogous effect on the solid bodies; but considering the tendency to the crystalization which then exist, the most natural structure gradually realised is the most stable for each particularly integrated molecule only, and by no means for the aggregate of these molecules which, on the contrary, tend to isolate from one another.

** These various relations result from the formulae (28), (35), (30), (10), (66), by calling x, y the actual coordinates of the molecules, l their actual distance to the free surface, ε a function of l. They subsist when, ε or $\rho gl/p$ being not constant, the relation (28^{bis}) is not satisfied; we shall see at Art. 42 how this can be arrived. In this case there are no more the coordinates of natural state $x-u$, $y-v$ varying with continuity from a particle to its neighbours; because the formula at the middle of page XXVII, equivalent to (28^{bis}) or (28^{ter}), expresses, as we know easily, that there exist two continuous functions u, v of x, y such that $\frac{du}{dx}$, $\frac{dv}{dy}$, $\frac{du}{dy}+\frac{dv}{dx}$ are equal to three given functions ∂_x, ∂_y, g_{xy}.

(83[ter])

$$K = cos\omega.$$

Again, let it be required to compute the *elastic* displacements which various points of the mass suffer with respect to the wall supposed to be immovable.

After the laws enunciated at page XXXIX, the elastic settling, apart from a motion of the aggregate of the whole pulverulent mass, are the displacements parallel to a fixed plane OM inclined by $\omega/2$ to the vertical, and equal to the product of the distance D of each particle to this plane by the constant factor $(sin\omega)/m$. But, after this fictitions movement, the material straight line emanating from the origin and initially inclined by ε_1 to the vertical, that is lying along the wall or making the angle $\varepsilon_1 - \omega/2$ with the direction of the displacements, makes with this a little less angle $\varepsilon_1 - \omega/2 - \zeta_2$. The small angle ζ_2, through which it has turned around the origin, is the such that the increment $Dcotg(\varepsilon_1 - \omega/2 - \zeta_2) - Dcotg(\varepsilon_1 - \omega/2)$ or sensibly $D\zeta_2/sin^2(\varepsilon_1 - \omega/2)$ of the distance of the origin O to the foot of the perpendicular D dropped from each of the points of the material line on the fixed plane under question is precisely equal to the displacement $D(sin\omega)/m$. As the value of the rotation ζ_2 which shall be given to each of the mass in the sense from Oy towards Ox after this fictitious settling to bring against the wall the couch which remains adjacent to it and thus obtain the definite situation of each of the mass, we have then

$$(84) \qquad \zeta_2 = \frac{sin\omega}{m} sin^2(\varepsilon_1 - \omega/2).$$

The angle $\zeta - \zeta_2$ which expresses the decrease, due to the *elastic* settling, of the inclination $\omega + \pi/2 - \varepsilon_1$ of the slope to the back face of the wall will be the excess over ζ_2 of the value (62) of ζ specified by $\varepsilon = \omega/2$. This angle then shall be equal to

$$\frac{sin\,\omega}{m}[cos^2\omega/2 - sin^2(\varepsilon_1 - \omega/2)]$$

$$= \frac{sin\,\omega}{m}[cos^2\omega/2 - cos^2\omega/2 sin^2\varepsilon_1 - sin^2\omega/2 cos^2\varepsilon_1 + 2 sin\omega/2 cos\omega/2 sin\varepsilon_1 cos\varepsilon_1]$$

$$= \frac{sin\omega cos\varepsilon_1 cos(\omega - \varepsilon_1)}{m},$$

and we shall have definitely, for the value of the reduction suffered by the angle of the top slope and the back face of the retaining wall,

$$(84^{bis}) \qquad \zeta-\zeta_2=\frac{sin\omega cos\varepsilon_1 cos(\omega-\varepsilon_1)}{m}.$$

38. *Case in Which the Equilibrium Produced Admits Only a Certain Degree of Stability.*

Now pass on to the case in which the retaining wall is not solid enough to enable the mass to acquire the maximum stability, and suppose ε constant so that a unique condition for the entire wall determines this parameter without that we have to find a special one for each value of l.

I shall admit that the wall behaves as a rigid body, movable more or less around a *fixed* axis parallel to that of z, in such a manner that its degree of firmness may be measured by the greatest value M which can be attained by the moment of the pressure around that axis without producing the overturning. Let b be the distance of the same axis to the back face of the wall, a the distance to the same axis of the perpendicular erected this back face at the one-third of its height L, the distance being reckoned positively above, and negatively below, the axis of rotation. The components, normal P' and tangential P'', of the total pressure P, shall have the respective moments aP', $-bP''$, and the total moment of the force which tends to produce the overturning of the wall will be equal to $aP'-bP''$, that is, after that which we have seen at the end of Art. 35 (p. LXX),

$$(85) \qquad \frac{\rho g L^2}{2}(aK'-bK'') = \frac{\rho g L^2}{2}\{[a-btg(\omega-\varepsilon_1)]K'-bsin\varepsilon_1\}.$$

In practice, the given a, b, ω, ε_1 shall have always the values which make this moment positive and increasing with K' when we shall vary the parameter ε characteristic of the equilibrium mode. As K', with the abstraction of the factor $cos(\omega-\varepsilon_1)$ independent of ε, represents the raito $-\mathfrak{R}/\rho gl$, whose derivative with respect to ε (form. 73) is negativ or positive according as ω is $>$ or $<$ zero, the moment (85) of the pressure will decrease indefinitely when ε changes from $\frac{\omega}{2}$ to $\frac{\omega\pm\tau}{2}$, by adopting in this last expression the upper or the lower sign according as the inclination ω shall be positive or negative. But $cos(\omega-2\varepsilon)$ will decrease at the same time from 1 to $cos\tau$, and ∂_1 after the first formula (60) will increase from $\pm(sin\omega)/m$ to $\pm(sin\omega)/(2mcos\tau)$. Thus the least admissible value of ∂_1 is that which corresponds to the value of ε, comprised between $\frac{\omega}{2}$ and $\frac{\omega\pm\tau}{2}$, for which the expression (85) is equal to the given moment M measuring the degree of firmness of the wall.

This value of ε defines the most stable equilibrium mode admitted by the circumstances, or consequently that which will be produced definitely. We shall obtain it by equating to M the second expression (85), which shall give the effective value of K' and consequently, after the second equation (82^{bis}), the value of K'' and that of the ratio $\{cos2(\varepsilon_1-\varepsilon)\}/\{cos2(\omega-\varepsilon)\}$. Let χ be the latter. We shall have then

$$cos2(\varepsilon_1-\varepsilon)=\chi cos2(\omega-\varepsilon),$$

or

$$cos2\varepsilon_1 cos2\varepsilon+sin2\varepsilon_1 sin2\varepsilon=\chi[cos2\omega cos2\varepsilon+sin2\omega sin2\varepsilon]:$$

whence we shall have

$$(85^{bis}) \qquad tg2\varepsilon=-\frac{cos2\varepsilon_1-\chi cos2\omega}{sin2\varepsilon_1-\chi sin2\omega},$$

the formula which makes known, without any indeterminateness, the angle 2ε comprised between ω and $\omega\pm\tau$, and always less than $\frac{\pi}{2}$ in its absolute value.

When once the parameter ε is determined, so shall be the equilibrium mode itself and the formulae (82) shall enable us to calculate the pressure. As to its moment with respect to the axis of rotation of the wall, it will be equal to M.

The equilibrium will become impossible and the wall overturned if the given constant M, which measures its power of resistance, were less than the value which the expression (85) assumes when we put therein for K' its least possible value, that is when ε becomes equal to $(\omega\pm\tau)/2$ and the mass passes to the state of collapse. At this moment K' or $-\{\mathfrak{N}cos(\omega-\varepsilon_1)\}/\rho gl$ or $\{\mathfrak{T}cos(\omega-\varepsilon_1)\}/\rho gl tg\phi_1$ receives the value

$$\frac{sin\phi cos(\omega-\varepsilon_1)cos\psi sin2(\varepsilon_1+\psi)}{2cos^2(\frac{\pi}{4}-\frac{\phi}{2})cos(\omega+\psi)tg\phi_1},$$

which results from the formulae (77) when we take therein for ψ the least (in absolute value) of the roots of the first equation (77). It is then necessary, that the wall does not overturn, that its power of resistance M may satisfy the inequality

$$(86) \qquad M>\frac{\rho gL^2}{2}\left\{-bsin\varepsilon_1+[a-btg(\omega-\varepsilon_1)]\frac{sin\phi cos(\omega-\varepsilon_1)cos\psi sin2(\varepsilon_1+\psi)}{2cos^2(\frac{\pi}{4}-\frac{\phi}{2})cos(\omega+\psi)tg\phi_1}\right\}.$$

All the values of M exceeding the second member of this inequality shall be compatible with an equilibrium mode, more or less stable, of the mass. The maximum of the internal stability will be attained when M shall have a value equal to or greater than that which the second expression (85) receives for $\varepsilon=\frac{\omega}{2}$.

In résumé, *when a retaining wall is not firm enough to permit the establishment of the equilibrium mode endowed with the maximum stability which corresponds to $\varepsilon=\omega/2$, the definite equilibrium which is really produced is that for which all the power of resistance of the wall is utilised.**

* If the earth mass had acquired cohesion in the long run, and if we admit that it is fixed in solidifying in the equilibrium mode possibly near to the natural state, that is in that for which the potential of elasticity Φ, equal to

$$\tfrac{1}{2}(\lambda+\mu)(\delta_1+\delta_3)^2+\tfrac{1}{2}\mu(\delta_1-\delta_3)^2$$

or to the quotient of

$$(\mu\delta_1-\mu\delta_3)^2+\frac{1}{\sin\phi}(\mu\delta_1+\mu\delta_3)^2$$

by 2μ, receives its smallest value, the formula (60^{bis}) (p. XXXIX) shall give, in substituting finally therein the value of $\cos(\omega-2\varepsilon)$ which results from (56^{bis}),

$$2\mu\,\Phi=p^2\left[\sin\phi+\frac{\sin^2\omega}{\cos^2(\omega-2\varepsilon)}\right]=(1+\sin\phi)\left(p-\frac{\rho gl\cos\omega}{1+\sin\phi}\right)^2+\rho^2g^2l^2\left(1-\frac{\cos^2\omega}{1+\sin\phi}\right).$$

The potential Φ, minimum for $p=\frac{\rho gl\cos\omega}{1+\sin\phi}$, is consequently the less the ratio $\frac{p}{\rho gl}$ is nearer the fraction $\frac{\cos\omega}{\cos^2\phi}(1-\sin\phi)$. This is besides less than the values of $\frac{p}{\rho gl}$ which correspond to the equilibrium modes, admissible only in a very deep mass and for which we have (note in p. LVII)

$$\cos^2(\omega-2\varepsilon)>\frac{\sin^2\omega}{\sin^2\phi};$$

in fact, the least of these values, calculated by means of (56^{bis}) (p. XXXIX) making therein

$$\cos^2(\omega-2\varepsilon)=\frac{\sin^2\omega}{\sin^2\phi}, \text{ is equal to } \frac{\cos\omega}{\cos^2\phi}\left(1-\sqrt{\frac{\sin^2\phi-\sin^2\omega}{1-\sin^2\omega}}\right)$$

consequently, this last value of $p/(\rho gl)$ is that which makes Φ as small as possible when the depth is very great. Thus, *the definitive equilibrium mode which will be produced in a very deep cohesive mass does not differ, as to the distribution of stresses, from the equilibrium limit which the same mass could show if it were pulverulent and of an angle of friction ϕ having its sine equal to $\frac{\mu}{\lambda+\mu}$: the lateral pressures are there minima.*

A less deep mass, on the contrary, allows the equilibrium modes for which the value of Φ is less: this value will be reduced even to its absolute minimum, if we can put $\frac{p}{\rho gl}=\frac{\cos\omega}{1+\sin\phi}$, without that the wall is overturned and without that the principal dilatation δ_1, then positive, attains on the deepest couch a value capable to determine the rupture.

A sufficiently less deep mass can also so dispose itself that there is at each point a plane element free from any stress among those which are normal to the plane of deformations. The relations (29) show that it happens, or that p_x, p_y vanish for a single value of β, only when $N_1N_2=T^2$, or, in view of the values (49^{bis}) (p. XXXIX) of N_1, N_2, T, when $\frac{\rho gl}{p}=2\cos\omega$; whence it follows, in supposing $\omega>0$, that the formula (56^{bis}) (p. XXXIX) gives $\omega-2\varepsilon=\frac{\pi}{2}+\omega$, $\varepsilon=\frac{\pi}{4}$. The formulae (57) show that $-\mathfrak{N}$, $\mathfrak{T}$ then vanish for $\varepsilon_1=0$, or that *the vertical plane elements suffer no stress. The equilibrium mode under consideration then agrees to the case of a mass cut vertically whose every vertical couch supports itself, without leaning upon its neighbours.*

These results and those which contain the note in p. LVII subsist independently from the formula (28^{ter}), that is without that we admit the existence of coordinates of natural state varying continuously from a particle to its neighbours. But when the effective depth of the mass is very great, $\lambda+\mu$ increases slowly with p (and more than μ) and it is necessary to suppose ϕ to be variable in such a manner that if the depth is excessive we shall have $\phi=0$ for very great l, and consequently $\omega=0$.

39. *More General Case in which the Moment of the Pressure Shall be Known Directly.*

The case in which we give the degree of stability of a retaining wall is then included in another more general, which is that in which we know the moment of the pressure with respect to the axis of rotation of the wall; it is one which happens when the wall, supposed to be weightless for the sake of simplicity, is maintained in equilibrium by means of a directly e-valuable constant force. Then the condition of equilibrium is obtained by equating the known moment M of this force to that (85) of the pressure, whence result as before the coefficients K', K'', and consequently the value of the parameter ε characteristic of the equilibrium mode. Only, the ex-pressions (85) being bound to be less or at most equal to the value which they take when $\varepsilon=\frac{\omega}{2}$, or when the equilibrium mode is the most stable pos-sible, ε will no more be comprised between $\frac{\omega}{2}$ and $\frac{\omega\pm\tau}{2}$ but between $\frac{\omega-\tau}{2}$ and $\frac{\omega+\tau}{2}$. It will be sufficient for this that the given moment M of the force charged to make an equilibrium to the pressure shall be comprised between the two values assumed by the second member of the inequality (86) when we put successively therein for ψ and ϕ_1 the two couples of values corresponding to the two modes of equilibrium limit spoken of in page LXV.

Admit, in conformity with that which always happens in practice, that the inclination ε_1 of the back face of the wall to the vertical is com-prised between $\frac{\omega}{2}\pm\frac{\pi}{4}$, or that in consequence the angle $\omega-2\varepsilon_1$ is less than $\frac{\pi}{2}$ in absolute value, and consider the elastic dilatation $\partial_{x'}$ suffered by the material lines of the mass initially normal to the same face. This dilatation measures the relative strain which is produced, to deviate from the natural state, between the back face of the wall and the material planes of the mass which were initially parallel to it and which are the such even after the displacements. The strain under question (negative or positive) is due partly to the weight itself of the mass and partly to the excess (positive or negative) of the stress which maintain the latter more or less compressed by the wall. The first formula (58) [p. XXXVIII] will give

$$(87) \qquad \partial_{x'}=\frac{sin\omega sin2(\varepsilon-\varepsilon_1)}{2m sin(\omega-2\varepsilon)}.$$

Differentiating this with respect to ε, and replacing, in the result

$cos(\omega-2\varepsilon)cos2(\varepsilon-\varepsilon_1)-sin(\omega-2\varepsilon)sin2(\varepsilon-\varepsilon_1)$ by $cos(\omega-2\varepsilon_1)$, we have

$$(87^{bis}) \qquad \frac{d\partial_{x'}}{d\varepsilon} = \frac{\sin\omega\cos(\omega-2\varepsilon_1)}{m\cos^2(\omega-2\varepsilon)}.$$

We see that $\partial_{x'}$ varies in the same or opposite sense as ε, according as ω is $>$ or $<$ o: in other words, $\partial_{x'}$ increases without limit when ε varies from $\frac{\omega\mp\tau}{2}$ to $\frac{\omega\pm\tau}{2}$.

This premised, conceive that the moment M of the pressure or of the external force which equilibrate it decreases from its upper to its lower limit so as to make ε vary from $\frac{\omega\mp\tau}{2}$ to $\frac{\omega\pm\tau}{2}$: the expression (87) of $\partial_{x'}$ will increase without limit. The formula (87) shows that it will vanish for $\varepsilon=\varepsilon_1$. Hence, *the various couches of pulverulent matter parallel to the back face of the wall are the nearer brought together and to this face, to deviate from the natural state, the greater the moment M of the pressure: they conserve their initial distances (one sliding before the other) if the moment of the pressure has such a value that $\varepsilon=\varepsilon_1$; they are brought nearer if the moment under question is greater than this value, and thrown asunder on the contrary if the moment of the pressure is less.*

At the time of an absolute immobility of the sustaining wall, the moment of the pressure assumes the value which corresponds to $\varepsilon=\frac{\omega}{2}$: then the degree of approaching or separating of the couches produced is due wholly to the weight itself of the mass; this is a mean state in which the wall tends neither to drive the mass behind it to upheave it and to make it retreat, nor to yield under its pressure by overturning forward. On the contrary, the greater values of the moment of the pressure correspond to the equilibrium states in which the wall, pressed by an external force against the mass, begins to drive it back by compressing it, while the smaller values correspond to the states in which the wall begins to yield under the pressure of the mass, by dilating in consequence forward.

All these modes of equilibrium are stable. In fact, we suppose the moment of the external force applied to the wall to equilibrate the pressure of the mass to be equal to a given constant. Now if the wall happens to leave its primitive position, either by separating from the mass or by approaching toward it, the latter will be loosened in the first, and compressed in the second, case, and the moment of its pressure against the wall will decrease or increase so as to be surpassed in the first case by the moment of the force charged to equilibrate it and to surpass it in the second; in consequence, the wall will certainly tend to resume its first position. Nevertheless, the stability of the equilibrium becomes incomplete when the moment of the pressure has exactly its greatest or least value possible, so that it cannot, according to the case, increase or decrease without that the rupture of the mass becomes inevitable. This, which we can call the *field of stability*, the interval τ for instance of the two limiting values $\frac{\omega \mp \tau}{2}$ of ε, reduces even to zero when $\omega = \pm \phi$: then the pressure can neither increase nor decrease without that the state of the mass becomes collapsing, and the equilibrium is unstable.

40. *Application to a Wall whose Back Face is Vertical.*

We see that it is by no means necessary to give to a sustaining wall a thickness which permits to establish the most stable equilibrium mode: it is sufficient, strictly speaking, for that this wall does not overturn and even resist the moderate shocks, that it can sustain a pressure a little greater than that which corresponds to the least stable equilibrium mode, the pressure given by the formulae (77) [p. LXV] in which we shall put for ψ its least value.

Suppose, for example, that the back face of the wall is vertical, the condition in virtue of which we have seen (form. 79) that the pressure of the mass, applied at the third of the height of the wall, becomes parallel to the top slope or makes with the horizon the constant inclination ω. The coefficient K, which enters in the expression $K\frac{\rho g L^2}{2}$ of this pressure, also represents the ratio $\frac{\Re cos(\omega - \varepsilon_1)}{\rho g l}$ now reduced to $\frac{\Re cos\omega}{\rho g l}$, and is equal to, after the first of (79),

$$(88) \qquad K = \frac{cos\omega cos 2\varepsilon}{cos 2(\omega - \varepsilon)}.$$

This value of K, then equal to $\frac{K'}{cos\omega}$, is reduced to $cos\omega$ for $\varepsilon = \frac{\omega}{2}$ and decreases constantly, as K', when ε comes from $\frac{\omega \mp \tau}{2}$ to $\frac{\omega \pm \tau}{2}$, at the second of these limits, the formula (79^{bis}) gives

$$(89) \qquad K - \frac{cos\omega sin 2\psi}{sin 2(\omega + \psi)},$$

ψ denoting the least, in absolute value, of the roots of the first equation in

(77).

This expression (89) of the least value of K becomes indeterminate when $\omega=0$; but it ceases to be the such, observing that, for the very small absolute values of ω, the first equation (77) can be replaced by $\omega+2\psi=\frac{\omega}{sin\phi}$ or $\frac{\psi}{\omega}=\frac{1-sin\phi}{2sin\phi}$, which gives

$$\frac{sin2\psi}{sin2(\omega+\psi)}=\frac{\psi}{\omega+\psi}=\frac{1-sin\phi}{1+sin\phi}=tg^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right).$$

The true value of K is then

(89bis) $$K=tg^2\left(\tfrac{\pi}{4}-\tfrac{\phi}{2}\right).$$

Let $\phi=45°$. The first equation (77) and the formula (89) admit to make the following table:

For $\omega=$	0,	$\pm10°$,	$\pm20°$,	$\pm30°$,	$\pm40°$,	$\pm45°$
$2\psi=$	0,	$\pm4°13'$,	$\pm8°56'$,	$\pm15°$,	$\pm25°22'$,	$\pm45°$
smallest value of $K=$	0·1716,	0·1765,	0·1935,	0·2320,	0·3404,	$\frac{1}{\sqrt{2}}=0·7071$;

while the values of K corresponding to the maximum stability of the state of the mass and equal to $cos\omega$ will be, for the same values of ω,

$K=1$, 0·9848, 0·9397, 0·8660, 0·7660, $\frac{1}{\sqrt{2}}=0·7071$.

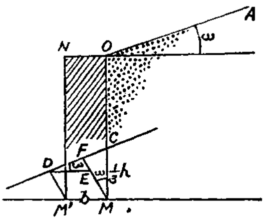
The equilibrium of the mass will then be stable provided the wall can sustain a pressure applied at the third of the height of its back face and directed parallelly to the top slope and a little greater than $K\frac{\rho g L^2}{2}$, K having the above values, 0·1716, 0·1765, etc, which correspond to the given declivity ω of the slope. The structure of the mass will acquire any possible stability if the wall could support, at the very point of application and along the same direction, a pressure equal to or greater than $K\frac{\rho g L^2}{2}$, K having the greatest values 1, 0·9848, etc lastly calculated

41. *Calculation of the Thickness to be Given to Such a Wall.*

Ordinarily, a sustaining wall $MONM'$ (fig. 4) tends to overturn by turning round the base M' of its front face, and we can suppose it to be maintained in equilibrium by its weight only. Designate, under these hypotheses, by ρ' its density, admit that its faces MO, $M'N$ are vertical, and find what equilibrium mode will be realised if we give it a thickness b sufficient to sustain itself but not great enough that the internal stability of the mass may be maximum.

After that which we have seen in p. LXXIX, it must be expressed that

Fig. 4.



the moment of the weight of the wall with respect to M' is equal to that lof he pressure P. This will be applied at C, at the third of the height h of the wall and equal to $K\frac{\rho g h^2}{2}$, K denoting the coefficient given by the formula (88); finally, it will be parallel to the top slope, that is directed along the straight line CD which makes the angle ω (always positive in the practice) with the horizon. We shall have its moment with respect to edge M' by multiplying it by the perpendicular $M'D$ dropped from M' on CD: now

$$M'D = MF - EF = \tfrac{1}{3}h\cos\omega - b\sin\omega;$$

the moment of the pressure is consequently

$$\tfrac{1}{2}\rho g h^2 K\left(\tfrac{1}{3}h\cos\omega - b\sin\omega\right).$$

On the other hand, the weight of the unit length of the wall is equal to $\rho' g b h$ and has for the arm, with respect to M', $\frac{1}{2}b$. Its moment is then equal to $\frac{1}{2}\rho' g h b^2$, and we have

$$\tfrac{1}{2}\rho g h^2 K\left(\tfrac{1}{3}h\cos\omega - b\sin\omega\right) = \tfrac{1}{2}\rho' g h b^2,$$

or, by dividing by $\frac{1}{2}\rho' g h^3$ and transposing,

(90) $$\left(\frac{b}{h}\right)^2 + \frac{\rho}{\rho'}K\sin\omega.\left(\frac{b}{h}\right) - \frac{\rho}{3\rho'}K\cos\omega = 0.$$

Such is the equation whence we shall have the value of K to substitute it in (88); this, solved with respect to $tg2\varepsilon$ after we have replaced therein $\cos 2(\omega-\varepsilon)$ by $\cos 2\varepsilon[\cos 2\omega + \sin 2\omega\, tg 2\varepsilon]$, will finally give the required value of ε.

But we can also solve equation (90) with respect to $\frac{b}{h}$, in order to know the value of the ratio $\frac{b}{h}$ of the thickness of the wall to its height which corresponds to some value of ε comprised between $\frac{\omega}{2}$ and $\frac{\omega \pm \tau}{2}$, that is to a more or less stable equilibrium mode. If we observe that $K\cos\omega$ is >0 and that the positive root only is proper, we find

$$\frac{b}{h} = -\frac{\rho}{2\rho'}K\sin\omega + \sqrt{\left(\frac{\rho}{2\rho'}K\sin\omega\right)^2 + \frac{\rho}{3\rho'}K\cos\omega},$$

or

$$(91) \quad \frac{b}{h}=\frac{\frac{\rho}{3\rho'}K\cos\omega}{\frac{\rho}{2\rho'}K\sin\omega+\sqrt{\left(\frac{\rho}{2\rho'}K\sin\omega\right)^2+\frac{\rho}{3\rho'}K\cos\omega}}$$

$$=\frac{2}{3}\,\frac{1}{tg\omega+\sqrt{tg^2\omega+\frac{4\rho'}{3\rho}\,\frac{1}{K\cos\omega}}}$$

For a determinate value of ω, this formula gives the ratio $\frac{b}{h}$ the less the coefficient K' itself the less, which shall be the case.

Suppose at first that the thickness of the wall is just sufficient for the equilibrium. Then K takes the values 0·1716, 0·1765, etc. given above, and if the density ρ' of the masonry is $\frac{3}{2}$ times of that, ρ, of the pulverulent mass, as it occurs ordinarily with a sufficient approximation, the ratio $\frac{b}{h}$ becomes

for $\omega=$	0°,	10°,	20°,	30°,	40°,	45°,
$\frac{b}{h}$(min. stability) =	0·1953	0·1866,	0·1802,	0·1761,	0·1786,	0 2060.

If, on the contrary, the thickness b has the value just neccessary for that the most stable equilibrium mode is produced, $K=\cos\omega$, and the formula (91) becomes

$$(92) \qquad \frac{b}{h}=\frac{2}{3}\,\frac{1}{tg\omega+\sqrt{tg^2\omega+\frac{4\rho'}{3\rho}(1+tg^2\omega)}}:$$

thus the ratio $\frac{b}{h}$ decreases without limit as ω increases, and we find, by putting always $\frac{\rho'}{\rho}=\frac{3}{2}$:

for $\omega=$	0°,	10°,	20°,	30°,	40°,	45°,
$\frac{b}{h}$(max. stability)=	0·4714,	0·4107,	0·3486,	0·2887,	0·2325,	0·2060.

For a given inclination ω of the slope to the horizon, the values of the ratio, $\frac{b}{h}$, of the thickness of a vertical sustaining wall to its height, which shall be equal or greater than the number registered in last table, will secure to the wall the greatest internal stability possible; the values less than the number given in last but one table will, on the contrary, be incompatible with the equilibrium, or too feeble that the wall does not begin to overturn; finally the intermediate values correspond to the various degrees of stability of the structure of the mass.

We find that the rule adopted in practice, and after which we give to a sustaining wall a thickness equal to the third of its height, offers a sufficient security, whenever the mass is not overcharged or is exposed to the negligible shocks only.

It is important to observe that the preceding formulae are applicable only when the depth of the earth mass is great enough, or at least uniform enough, that, in the regions neighbouring the sustaining wall, the stresses may be sensibly equal everywhere at the same distance from the top slope. We then neglect, by employing them, the disturbing influence exerted by the ground beneath when it is not parallel to the free surface of the mass. This influence must be insensible in the ordinary circumstances of practice; because the inclination ω being there positive, the pressure, transmitted from above downward parallelly to the top slope proceeds to the earth couches really deep. But it will not be the same if the inclination ω is negative and the mass is, at some distance from the wall, of an insignificant thickness only.