ON THE SURFACES OF SKEW ARCHES.

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The heading surfaces, the coursing surfaces and the soffit of skew arches are, as we know, not mutually orthogonal, and the question naturally arises when and how the conditions of their orthogonalism can be satisfied. The following considerations solve this problem.

Taking the rectangular axes x, y, z in which z–axis is parallel to the element of the soffit, let

$$\xi(x, y, z)=\text{const.},\quad \eta(x,y,z)=\text{const.},\quad \zeta(x,y)=\text{const.}$$

denote the heading surfaces, the coursing surfaces and the soffit respectively. The necessary and sufficient conditions that these surfaces are mutually orthogonal are

$$1)\qquad \begin{cases} \dfrac{\partial\eta}{\partial x}\dfrac{\partial\zeta}{\partial x}+\dfrac{\partial\eta}{\partial y}\dfrac{\partial\zeta}{\partial y}=0, \\ \dfrac{\partial\zeta}{\partial x}\dfrac{\partial\xi}{\partial x}+\dfrac{\partial\zeta}{\partial y}\dfrac{\partial\xi}{\partial y}=0, \\ \dfrac{\partial\xi}{\partial x}\dfrac{\partial\eta}{\partial x}+\dfrac{\partial\xi}{\partial y}\dfrac{\partial\eta}{\partial y}+\dfrac{\partial\xi}{\partial z}\dfrac{\partial\eta}{\partial z}=0, \end{cases}$$

whose first two equations give

$$\frac{\dfrac{\partial\xi}{\partial y}}{\dfrac{\partial\xi}{\partial x}}=\frac{\dfrac{\partial\eta}{\partial y}}{\dfrac{\partial\eta}{\partial x}}=-\frac{\dfrac{\partial\zeta}{\partial x}}{\dfrac{\partial\zeta}{\partial y}}=F(x,y),\ \text{say}.$$

Thus we find that

$$\begin{cases} \dfrac{\partial\xi}{\partial x}=f(z)\varphi(x,y), & \dfrac{\partial\xi}{\partial y}=f(z)\varphi(x,y)F(x,y), & \dfrac{\partial\xi}{\partial z}=\sigma(x,y,z), \\ \dfrac{\partial\eta}{\partial x}=g(z)\psi(x,y), & \dfrac{\partial\eta}{\partial y}=g(z)\psi(x,y)F(x,y), & \dfrac{\partial\eta}{\partial z}=\tau(x,y,z), \end{cases}$$

and in order that

$$\frac{\partial\xi}{\partial x}dx+\frac{\partial\xi}{\partial y}dy+\frac{\partial\xi}{\partial z}dz,\qquad \frac{\partial\eta}{\partial x}dx+\frac{\partial\eta}{\partial y}dy+\frac{\partial\eta}{\partial z}dz$$

may be total differentials, we must have

$$\begin{cases} \dfrac{\partial\varphi}{\partial y}=\varphi\dfrac{\partial F}{\partial x}+F\dfrac{\partial\varphi}{\partial x}, & f'\varphi F=\dfrac{\partial\sigma}{\partial y}, \quad f'\varphi=\dfrac{\partial\sigma}{\partial x}, \\ \dfrac{\partial\psi}{\partial y}=\psi\dfrac{\partial F}{\partial x}+F\dfrac{\partial\psi}{\partial x}, & g'\psi F=\dfrac{\partial\tau}{\partial y}, \quad g'\psi=\dfrac{\partial\tau}{\partial x}. \end{cases}$$

Putting then

$$\int(\varphi dx+\varphi F dy)=\Phi(x,y), \quad \int(\psi dx+\psi F dy)=\Psi(x,y),$$

we find

$$\xi=f\Phi+\alpha(z), \qquad \eta=g\Psi+\beta(z).$$

Substituting these values of ξ and η in the third equation of 1), we have

$$fg\varphi\psi(1+F^2)+(f'\Phi+\alpha')(g'\Psi+\beta')=0,$$

so that

$$-g'\Psi=\beta'+\frac{fg\varphi(1+F^2)}{f'\Phi+\alpha'}\psi,$$

whence

$$-g'\frac{\partial\Psi}{\partial x}=-g'\psi=\psi\frac{\partial}{\partial x}\left[\frac{fg\varphi(1+F^2)}{f'\Phi+\alpha'}\right]+\frac{fg\varphi(1+F^2)}{f'\Phi+\alpha'}\frac{\partial\psi}{\partial x},$$

$$-g'\frac{\partial\Psi}{\partial y}=-g'\psi F=\psi\frac{\partial}{\partial y}\left[\frac{fg\varphi(1+F^2)}{f'\Phi+\alpha'}\right]+\frac{fg\varphi(1+F^2)}{f'\Phi+\alpha'}\frac{\partial\psi}{\partial y}.$$

Putting

$$\psi\frac{\partial F}{\partial x}+F\frac{\partial\psi}{\partial y}$$

instead of $\partial\psi/\partial y$ in the second equation, and equating the values of $\partial\psi/\partial x$ from the two equations we shall find that we must have

$$\varphi\left(\frac{\partial F}{\partial x}+F\frac{\partial F}{\partial y}\right)=0.$$

If

$$\frac{\partial F}{\partial x}+F\frac{\partial F}{\partial y}=0,$$

then F must be a constant and consequently ζ–surface a plane, which is of no use in the present case. If $\varphi=0$, then ξ is independent of x and y, and is a function of z only, so that ξ–surface is a plane parallel to the co-ordinate plane xy, η–surface being giveu by

$$\frac{\partial\eta}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\eta}{\partial y}\frac{\partial\zeta}{\partial y}=0, \qquad \frac{\partial\eta}{\partial z}=0,$$

showing that it is a cylindrical surface whose elements are parallel to z–axis or those of ζ–surface. The surfaces determined by

$$\zeta+i\eta=h(x+iy)$$

where h is a monogenic function are examples of η– and ζ–surfaces. Such three surfaces ξ–, η–, ζ–surfaces can hardly be used in skew arches, and we conclude that the required conditions on the orthogonalism of the three surfaces in question cannot be satisfied with such kind of arches.

In fact, in order that ξ–, η–, ζ–surfaces may be orthogonal, we see that, by Dupin's theorem, the lines of intersection of the first two with the latter are the lines of curveture of the latter and therefore given by

$$\begin{vmatrix} \dfrac{dx}{ds}\dfrac{\partial^2\zeta}{\partial x^2}+\dfrac{dy}{ds}\dfrac{\partial^2\zeta}{\partial x\partial y} & \dfrac{dx}{ds} & \dfrac{\partial\zeta}{\partial x} \\ \dfrac{dx}{ds}\dfrac{\partial^2\zeta}{\partial x\partial y}+\dfrac{dy}{ds}\dfrac{\partial^2\zeta}{\partial y^2} & \dfrac{dy}{ds} & \dfrac{\partial\zeta}{\partial y} \\ 0 & \dfrac{dz}{ds} & 0 \end{vmatrix}=0, \quad \zeta=\text{const.}$$

The first determinant equation can easily be shown to be reduced to the equation

$$dzd\left\{\frac{\dfrac{\partial\zeta}{\partial y}}{\dfrac{\partial\zeta}{\partial x}}\right\}=0.$$

Thus the lines of curveture are given by

$$\begin{cases} z=\text{const.}, \\ \zeta=\text{const.}, \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial\zeta}{\partial y}=a\dfrac{\partial\zeta}{\partial x}, \\ \zeta=\text{const.}, \end{cases}$$

the first of which is the line of intersection with ξ–surface, and the latter, the element of ζ–surface, that with η–surface.