§ VII.

CALCULATION OF THE PRESSURE EXERTED ON EACH ELEMENT OF SURFACE NORMAL TO THE PLANE OF DEFORMATIONS, AND OF THE TOTAL PRESSURE SUPPORTED BY A PLANE SUSTAINING WALL.

31. *Formulae which give the Direction ϕ_1 and the Value $\mathfrak{R}$ of the Pressure* $(-\mathfrak{N},\ \mathfrak{T})$.

It remains us to study the pressure, having for normal and tangential components $-\mathfrak{N}$, $\mathfrak{T}$, and supported by a unit area of each plane element parallel to the axis of z, and then to evaluate the total pressure suffered by any plane section taken through this axis in the mass : this pressure will evidently be exerted on a plane sustaining wall whose back face will coincide with the section under consideration, when the mass, instead of being infinite, would be reduced to its part situated at the back of such a wall, while the contiguous pulverulent couch had been in the same time undisturbed in the positions $x+u$, $y+v$, $z+w$ which it occupies in the infinite mass.

Let ε_1 be the inclination to the vertical of any plane element parallel to the axis of z. The two components $-\mathfrak{N}$ and $\mathfrak{T}$ of the pressure which it receives on unit area are, after the relations (57) [p. XXXVII],

$$(72)\quad \begin{cases} -\mathfrak{N} = \dfrac{\rho g l}{cos2(\omega-\varepsilon)}[cos(\omega-2\varepsilon)+sin\omega sin2(\varepsilon_1-\varepsilon),] \\ \mathfrak{T} = \dfrac{\rho g l}{cos2(\omega-\varepsilon)} sin\ \omega\ cos2(\varepsilon_1-\varepsilon). \end{cases}$$

I will call

$$\mathfrak{R}$$

the total pressure whose two components, normal and tangential, are $-\mathfrak{N}$ and $\mathfrak{T}$,

$$\phi_1$$

the angle, comprised at most between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, which it makes with the prolongation of the normal to the plane element and whose tangent, $\dfrac{\mathfrak{T}}{-\mathfrak{N}}$, define it completely. We shall have

$$tg\phi_1=\frac{\mathfrak{T}}{-\mathfrak{N}},\quad \mathfrak{R}=\frac{\mathfrak{T}}{sin\phi_1},$$

or

$$(72^{bis})\quad tg\phi_1=\frac{sin\omega cos2(\varepsilon_1-\varepsilon)}{cos(\omega-2\varepsilon)+sin\omega sin2(\varepsilon_1-\varepsilon)},\quad \mathfrak{R}=\frac{sin\omega cos2(\varepsilon_1-\varepsilon)}{cos2(\omega-\varepsilon)sin\phi_1}\rho gl.$$

The first of these formulae presents the inconvenience of not being calculable by logarithms. It is preferable to evaluate $tg(\phi_1+\varepsilon_1-\varepsilon\pm\frac{\pi}{4})$ by means of the relation

$$tg(\phi_1+\varepsilon_1-\varepsilon\pm\tfrac{\pi}{4})=\frac{tg\phi_1+tg(\varepsilon_1-\varepsilon\pm\frac{\pi}{4})}{1-tg\phi_1 tg(\varepsilon_1-\varepsilon\pm\frac{\pi}{4})},$$

in the second member of which we shall put for $tg\phi_1$ its value (72^{bis}). This second member will become much more simple if we replace therein

$cos2(\varepsilon_1-\varepsilon)$ or $\pm sin2(\varepsilon_1-\varepsilon\pm\frac{\pi}{4})$ by $\pm 2cos(\varepsilon_1-\varepsilon\pm\frac{\pi}{4})sin(\varepsilon_1-\varepsilon\pm\frac{\pi}{4})$,

$sin2(\varepsilon_1-\varepsilon)$ or $\pm cos2(\varepsilon_1-\varepsilon\pm\frac{\pi}{4})$ by $\mp cos^2(\varepsilon_1-\varepsilon\pm\frac{\pi}{4})\pm sin^2(\varepsilon_1-\varepsilon\pm\frac{\pi}{4})$.

We thus find

$$(72^{ter})\quad tg(\phi_1+\varepsilon_1-\varepsilon\pm\tfrac{\pi}{4})=tg(\varepsilon_1-\varepsilon\pm\tfrac{\pi}{4})\frac{cos(\omega-2\varepsilon)\pm sin\omega}{cos(\omega-2\varepsilon)\mp sin\omega}$$
$$=tg(\varepsilon_1-\varepsilon\pm\tfrac{\pi}{4})\frac{tg\frac{1}{2}(\frac{\pi}{2}-\omega+2\varepsilon\pm\omega)}{tg\frac{1}{2}(\frac{\pi}{2}-\omega+2\varepsilon\mp\omega)}.$$

This formula, in which we can take upper or lower sign at our pleasure, will make the angle ϕ_1 known without any indetermination, which fixes the direction of the pressure, when we shall have given the inclination ω of the top slope to the horizon, the angle ε_1 made by the plane element with the vertical and the parameter ε characteristic of the equilibrium mode. The second relation (72^{bis}) will permit us to calculate in consequence by logarithms the magnitude $\mathfrak{R}$ of the pressure per unit area, if we know besides the depth l at which the plane element is situated, that is its distance normal to the top slope and the weight ρg of the apparent unit volume of the mass.

We obtain between $\mathfrak{R}$ aud $\mathfrak{T}$ an extremely simple relation by adding the two equations (72) after having multiplied them respectively by $-sin(\omega-\varepsilon_1)$ and $cos(\omega-\varepsilon_1)$, then replacing in the second member the terms affected by $sin\omega$ by their total value

$$sin\omega cos(\omega-2\varepsilon+\varepsilon_1)=sin\omega[cos(\omega-2\varepsilon)cos\varepsilon_1-sin(\omega-2\varepsilon)sin\varepsilon_1],$$

and similarly the term $-sin(\omega-\varepsilon_1)cos(\omega-2\varepsilon)$ by

$$cos(\omega-2\varepsilon)[-sin\omega\, cos\varepsilon_1+cos\omega\, sin\varepsilon_1],$$

and lastly by reducing. It thus gives

$$(72^{quater})\qquad \mathfrak{R}\, sin(\omega-\varepsilon_1)+\mathfrak{T}cos(\omega-\varepsilon_1)=\rho gl\, sin\varepsilon_1.$$

This formula is nothing but a particular application of the theorem, called one *of reciprocity*, expressed by the relations (21) [p. XXI]: its first member represents the projection, on the normal to the top slope, of the elastic force suffered by the plane element of the inclination ε_1, and the second member is the projection, on the normal to this plane element, of the elastic force supported by another plane element passing through the same point and parallel to the top slope: the theorem above cited shows that these two projections must be equal.

32. *How the Two Components* $-\mathfrak{N}$, $\mathfrak{T}$ *Vary with* ε.

To find how vary the two components $-\mathfrak{N}$, $\mathfrak{T}$ of the pressure $\mathfrak{N}$ exerted on a same plane element, when we make the equilibrium mode change, that is when we make the angular parametre ε vary.

After the relation (70^{bis}), it will be sufficient to make ε increase from $\frac{\omega-\tau}{2}$ to $\frac{\omega+\tau}{2}$, or to make $2(\omega-\varepsilon)$ decrease from $\omega+\tau$ to $\omega-\tau$, that is from an upper limit less than $\frac{\pi}{2}$ to a lower limit greater than $-\frac{\pi}{2}$. In this interval, the expressions (72) of $-\mathfrak{N}$, $\mathfrak{T}$ remain constantly finite and continuous. The denominator $cos2(\omega-\varepsilon)$ vanishes at the two limits in the particular case $\omega=0$, $\tau=\frac{\pi}{2}$; but the numerator vanishes at the same time and the values of $-\mathfrak{N}$, $\mathfrak{T}$ taken at these two limits tend, in proportion as ω decreases up to zero, toward the perfectly determinate values given by the formulae (77) in sequel.

Differentiate then, with respect to ε, the expression (72) of $-\mathfrak{N}$, $\mathfrak{T}$ and replace in the results

$sin2(\omega-\varepsilon)cos(\omega-2\varepsilon)-cos2(\omega-\varepsilon)sin(\omega-2\varepsilon)$ by $sin[2(\omega-\varepsilon)-(\omega-2\varepsilon)]=sin\omega$,

$cos2(\omega-\varepsilon)cos2(\varepsilon_1-\varepsilon)+sin2(\omega-\varepsilon)sin2(\varepsilon_1-\varepsilon)$ by $cos2(\omega-\varepsilon_1)$;

$sin2(\omega-\varepsilon)cos2(\varepsilon_1-\varepsilon)-cos2(\omega-\varepsilon)sin2(\varepsilon_1-\varepsilon)$ by $sin2(\omega-\varepsilon_1)$:

we shall have simply

$$(73)\qquad \begin{cases} \dfrac{d(-\mathfrak{N})}{d\varepsilon}=\dfrac{-2\rho gl\, sin\omega[1+cos2(\omega-\varepsilon_1)]}{cos^2 2(\omega-\varepsilon)}, \\ \dfrac{d\mathfrak{T}}{d\varepsilon}=\dfrac{-2\rho gl\, sin\omega sin2(\omega-\varepsilon_1}{cos^2 2(\omega-\varepsilon)}. \end{cases}$$

These derivatives of $-\mathfrak{N}$ and $\mathfrak{T}$ have constantly the same signs whatever ε may be: the first is negative or positive according as the inclination ω to the horizon of the top slope is in itself positive or negative; the second has the same sign as the first when the sine of the angle $2(\omega-\varepsilon_1)$ is positive, and the opposite sign in the case when this sine is negative. *Each of the two components* $-\mathfrak{N}$, $\mathfrak{T}$ *then varies continuously in the same sense when* ε *increases, and it is comprised at each instant between the two extreme values which it receives for* $\varepsilon=\frac{\omega\mp\tau}{2}$: *also if we wish to have the least value of the normal pressure* $-\mathfrak{N}$, *we must take* $\varepsilon=\frac{\omega+\tau}{2}$ *or* $\varepsilon=\frac{\omega-\tau}{2}$ *according as* ω *is* $>$ *or* $<$ *zero.*

The second formula (72) could be deduced from the first and the relation (72^{quater}), which gives $\mathfrak{T}$ as a linear function of $\mathfrak{N}$, no coefficient of which function depends on ε, and after which the ratio of the two derivatives $\frac{d\mathfrak{T}}{d\varepsilon}$ and $\frac{d(-\mathfrak{R})}{d\varepsilon}$ is equal to $tg(\omega-\varepsilon_1)$.

33. *Extreme Values of the Components $-\mathfrak{R}$, $\mathfrak{T}$. Study of the Limiting Equilibrium which can be Presented by the Infinite Mass*

To evaluate the extreme values of $-\mathfrak{R}$, and at first those which correspond to $\varepsilon=\frac{\omega+\tau}{2}$ for positive ω, to $\varepsilon=\frac{\omega-\tau}{2}$ for negative ω, and consequently to the least value of $-\mathfrak{R}$.

I will call ψ the auxiliary angle, comprised between o and $\frac{\pi}{4}-\frac{\omega}{2}$ when ω is $>$o, between o and $-\frac{\pi}{4}-\frac{\omega}{2}$ when ω is $<$o, and defined by the equation

$$(74)\qquad sin(\omega+2\psi)=\frac{sin\omega}{sin\phi}.$$

This equation, compared to (70), shows that τ is equal to the complement of the absolute value of $\omega+2\psi$, or that we have

$$(74^{bis})\qquad \tau=\frac{\pi}{2}\mp(\omega+2\psi),$$

and consequently

$$(74^{ter})\qquad -2\varepsilon \text{ or } -(\omega\pm\tau)=\mp\frac{\pi}{2}+2\psi,$$

$$2(\omega-\varepsilon)=\mp\frac{\pi}{2}+2(\omega+\psi),\ 2(\varepsilon_1-\varepsilon)=\mp\frac{\pi}{2}+2(\varepsilon_1+\psi),\ \omega-2\varepsilon=\mp\frac{\pi}{2}+(\omega+2\psi).$$

The formulae (72) become, if we put therein these values of $2(\omega-\varepsilon)$, $2(\varepsilon_1-\varepsilon)$, $\omega-2\varepsilon$,

$$(75)\quad \mathfrak{T}=\frac{sin\omega\, sin2(\varepsilon_1+\psi)}{sin2(\omega+\psi)}\rho gl,\ -\mathfrak{R}=\frac{sin(\omega+2\psi)-sin\omega\, cos2(\varepsilon_1+\psi)}{sin2(\omega+\psi)}\rho gl.$$

The expression (75) of $\mathfrak{T}$ appears indeterminate when $\omega=$o. We can transform it by making successively

$$(75)^{bis}\quad sin2(\omega+\psi)=2sin(\omega+\psi)cos\psi\frac{cos(\omega+\psi)}{cos\psi}=$$

$$[sin(\omega+\psi-\psi)+sin(\omega+\psi+\psi)]\ \frac{cos(\omega+\psi)}{cos\psi},$$

and then replacing therein the ratio

$$\frac{sin\omega}{sin\omega+sin(\omega+2\psi)},\ \text{or after (74)},\ \frac{sin\phi sin(\omega+2\psi)}{sin\phi sin(\omega+2\psi)+sin(\omega+2\psi)},$$

by

$$\frac{sin\phi}{1+sin\phi}=\frac{sin\phi}{2cos^2(\frac{\pi}{4}-\frac{\phi}{2})}.$$

We shall have thus

$$(76)\qquad \mathfrak{T}=\frac{\rho gl\, sin\phi}{2cos^2(\frac{\pi}{4}-\frac{\phi}{2})}\cdot\frac{cos\psi sin2(\varepsilon_1+\psi)}{cos(\omega+\psi)}.$$

As to the expression (75) of $-\mathfrak{R}$, it is not calculable by logarithms;

but we shall obtain $-\mathfrak{N}$ when $\mathfrak{T}$ will have been evaluated by means of (76), and if we know the angle ϕ_1 which measures the inclination of the resultant pressure $\mathfrak{R}$, applied to the plane element under question, to the prolongation of the normal to this plane element. The formula (72^{ter}) will give, for the calculation of this angle, when we make after (74^{ter}) $-\varepsilon = \mp\frac{\pi}{4} + \psi$ and also $cos(\omega - 2\varepsilon) = \pm sin(\omega + 2\psi) = \pm \frac{sin\omega}{sin\phi}$,

$$(76^{bis}) \quad tg(\phi_1 + \varepsilon_1 + \psi) = tg(\varepsilon_1 + \psi)\ \frac{1 + sin\phi}{1 - sin\phi} = \frac{tg(\varepsilon_1 + \psi)}{tg^2(\frac{\pi}{4} - \frac{\phi}{2})}.$$

Finally we shall have

$$-\mathfrak{N} = \frac{\mathfrak{T}}{tg\phi_1}.$$

Now to evaluate another extreme value of $-\mathfrak{N}$ and $\mathfrak{T}$, knowing the values which correspond to $\varepsilon = \frac{\omega - \tau}{2}$ for $\omega > 0$ and to $\varepsilon = \frac{\omega + \tau}{2}$ for $\omega < 0$. It is convenient to take, instead of the root ψ of equation (74) which is comprised between 0 and $\pm\frac{\pi}{4} - \frac{\omega}{2}$, ψ' which is comprised between $\pm\frac{\pi}{4} - \frac{\omega}{2}$ and $\pm\frac{\pi}{2} - \frac{\omega}{2}$, or which is such that, with respect to the preceding ψ,

$$(76^{ter}) \qquad \omega + 2\psi' = \pm\pi - (\omega + 2\psi),$$

adopting the + or − sign according as ω is $>$ or $<$ zero. We shall have, instead of (74^{bis}),

$$\tau = -\frac{\pi}{2} \pm (\omega + 2\psi'),$$

and consequently, by virtue of $\varepsilon = \frac{\omega \mp \tau}{2}$,

$$-2\varepsilon = \mp\frac{\pi}{2} + 2\psi'.$$

This value of -2ε differs from the preceding (74^{ter}) in this only that ψ' replaces ψ. Hence, aside the change of ψ to ψ', the formulae (72) lead to the same relations (75), (76), (76^{bis}) as in the preceding case.

To sum up, *the extreme values of the two components* $-\mathfrak{N}$, $\mathfrak{T}$ *of the pressure exerted on a first plane element making the angle* ε_1 *with the vertical that is those which correspond to the two modes of equilibrium limit allowed by an infinite mass inclined by* ω *to the horizon, are given by the formulae*

$$(77) \qquad \begin{cases} sin(\omega + 2\psi) = \dfrac{sin\omega}{sin\phi}, \quad tg(\phi_1 + \varepsilon_1 + \psi) = \dfrac{tg(\varepsilon_1 + \psi)}{tg^2(\frac{\pi}{4} - \frac{\phi}{2})}, \\ \mathfrak{T} = \dfrac{sin\phi cos\psi sin2(\varepsilon_1 + \psi)}{2cos^2(\frac{\pi}{4} - \frac{\phi}{2})cos(\omega + \psi)}\rho gl, \quad -\mathfrak{N} = \dfrac{\mathfrak{T}}{tg\phi}; \end{cases}$$

the auxiliary angle ψ *which can be calculated by the first equation (77) should be so chosen that the sum* $\omega + 2\psi$ *may be, in absolute value, less than* $\frac{\pi}{2}$ *for the equilibrium mode which gives the normal component* $-\mathfrak{N}$ *of the pressure its least value, and comprised, on the contrary, between* $\frac{\pi}{2}$ *and* π, *or supplementary to the preceding, for the equilibrium mode which gives* $-\mathfrak{N}$ *its great-*

est value; as to the inclination ϕ_1 *of the pressure to the prolongation of the normal to the plane element, as it can vary at most between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ *only, the second equation* (77) *will completely determine it by knowing* $tg(\phi_1+\varepsilon_1+\psi)$.

The angle ψ has an important geometrical signification. We have

$$(77^{bis})\qquad -2\varepsilon=\mp\tfrac{\pi}{2}+2\psi,\ \text{or}\ -\varepsilon=\mp\tfrac{\pi}{4}+\psi,$$

and consequently, after (59) [p. XXXVIII], ε' (or $\varepsilon\mp\frac{\pi}{4}$)$=-\psi$. Now ε' denotes the angle made with the vertical, on the side of OA (fig. 1, p. XXXVI), by one of the two principal dilatations or of the two principal stresses produced at each point: ψ then represents the same angle, but estimated positively in the opposite sense, that is on the side of Oy.

This is nevertheless what we recognize by putting successively $\varepsilon_1=-\psi$, $\varepsilon_1=-\psi+\frac{\pi}{2}$ in the formulae (75). In these two cases $\mathfrak{T}=0$, and if $\mathfrak{N}_1$, $\mathfrak{N}_2$ denote the two values of $\mathfrak{N}$,

$$-\mathfrak{N}_1=\frac{sin(\omega+2\varphi)-sin\omega}{sin2(\omega+\psi)}\rho gl,\qquad -\mathfrak{N}_2=\frac{sin(\omega+2\psi)+sin\omega}{sin2(\omega+\varphi)}\rho gl.$$

Sin $(\omega+2\psi)$ and $sin\omega$ having always same signs after (74), $-\mathfrak{N}_1$ is the least of the principal stresses, and $-\mathfrak{N}_2$ the greatest. Consequently *the direction which makes with the vertical the angle* $-\psi$ *coincides with the profil of the plane element on which the smallest principal stress is exerted.*

The above formulae of $-\mathfrak{N}_1$, $-\mathfrak{N}_2$ are simplified when we replace $sin(\omega+2\psi)-sin\omega$ by $2cos(\omega+\psi)\,sin\psi$, $sin(\omega+2\psi)+sin\omega$ by $2sin(\omega+\psi)cos\psi$, $sin2(\omega+\psi)$ by $2sin(\omega+\psi)cos(\omega+\psi)$: they become

$$(77^{ter})\qquad -\mathfrak{N}_1=\frac{sin\psi}{sin(\omega+\psi)}\rho gl,\quad -\mathfrak{N}_2=\frac{cos\psi}{cos(\omega+\psi)}\rho gl.$$

34. *Stress sustained by a Vertical Plane Element.*

The mode of equilibrium limit for which the component $-\mathfrak{N}$ of the stress acquires its least possible value is precisely that which has been studied by Mr. Maurice Levy in his memoir *sur une théorie rationnelle de l'équilibre des terres fraichement remuées et ses applications an calcul de la stabilité des murs de soutènement.** The formulae (77) do not differ from those which have been given at Art. 15 of this memoir.

Mr. Macquorn-Rankine had already, since 1856, considered the two modes of equilibrium limit which can be exhibited by a mass limited by a plane slope and about to collapse in all its extent: he had set forward the laws which govern the pressure exerted on vertical plane elements, that is

* This work, presented for the first time to l'Académie des sciences de Paris at June 3, 1867 and reproduced at June 21, 1869, is found in *Journal de Mathématique de M. Liouville* (t. XVIII, 1873).

on those for which $\varepsilon_1=0$.*

These laws are very simple even when the equilibrium is not limiting. In fact, putting $\varepsilon_1=0$ in the general formulae (72), we shall have

$$(78)\qquad \mathfrak{T}=\frac{cos2\varepsilon sin\omega}{cos2(\omega-\varepsilon)}\rho gl,\quad -\mathfrak{R}=\frac{cos2\varepsilon cos\omega}{cos2(\omega-\varepsilon)}\rho gl,$$

whence we have evidently to take, for respective values of the resultant pressure $\mathfrak{R}$ and of its inclination ϕ_1 to the prolongation of the normal to the vertical plane element under consideration,

$$(79)\qquad \mathfrak{R}=\frac{cos2\varepsilon}{cos2(\omega-\varepsilon)}\rho gl,\quad \phi_1=\omega.$$

The second relation (79) signifies that, *in an infinite mass, each vertical plane element suffers a pressure parallel to the top slope.* The first, specified for the modes of equilibrium limit, that is, for the values of -2ε equal to $\mp\frac{\pi}{2}+2\psi$, gives

$$(79^{bis})\qquad \mathfrak{R}=\frac{sin2\psi}{sin2(\omega+\psi)}\rho gl.$$

Now we can put

$sin2\psi$ or $sin(\omega+2\psi-\omega)$ by $sin(\omega+2\psi)cos\omega-cos(\omega+2\psi)sin\omega$,

$sin2(\omega+\psi)$ or $sin(\omega+2\psi+\omega)$ by $sin(\omega+2\psi)cos\omega+cos(\omega+2\psi)sin\omega$,

and observe in consequence that, after (74),

$$sin(\omega+2\psi)=\frac{sin\omega}{sin\phi},\ cos(\omega+2\psi)=\frac{\pm\sqrt{sin^2\phi-sin^2\omega}}{sin\phi}=\frac{\pm\sqrt{cos^2\omega-cos^2\phi}}{sin\phi},$$

the upper signs corresponding to the absolute value of $\omega+2\psi$ less than $\frac{\pi}{2}$, or to the equilibrium mode for which the normal pressures are the least possible, and the lower signs corresponding to the other mode. It thus becomes definitely

$$(80)\qquad \mathfrak{R}=\frac{cos\omega\mp\sqrt{cos^2\omega-cos^2\phi}}{cos\omega\pm\sqrt{cos^2\omega-cos^2\phi}}\rho gl.$$

The upper signs give the value of the most feeble stress exerted on the vertical plane element which is produced when the mass is about to collapse downward and the friction of the earth acts with the possible maximum to retain it, the lower signs give on the contrary the greatest value

* See in the *Annales des ponts et chaussées* (November, 1872, p. 242) a note in which Mr. Flamant, ingénieur des ponts et chaussées, has given a very simple geometrical exposition of the theory of Mr. Rankine. Mr. Considère has also inserted, at June, 1870 (pp. 545 to 594) in these *Annales*, a memoir containing, besides the same results at which he had arrived from his own part, many judicious and interesting considerations.

of the pressure which it exhibits, as Messrs. Considère and Flamant have remarked, when the same mass, compressed horizontally, is supposed to be at the point of collapsing upward (or rather of flowing back above its free surface), and consequently when the friction balances to an increase of the weight of the mass: it is this kind of pressure which Poncelet has designated by the name of *abuting of the earths* (*butée des terres*).

We observe that *the two modes of equilibrium limit correspond to the two kinds of collapsing, by pulling and by crushing or compression, which can be exhibited by a pulverulent mass with the plane top face, when it is disturbed at the same time through its whole extent.*

35. *Calculation of the Total Pressure Suffered by a Wall with Plane Back Face.*

Lastly to calculate the total stress supported per unit horizontal length of a plane section taken in the mass through the axis of z up to any distance L from this axis and also having to the vertical any inclination ε_1. This pressure will evidently be that which is undergone by the back face of wall contiguous to the mass along the section in question and supposed to be capable to produce on the pulverulent mass situated at its back the same effect as produced, in the infinite mass, by the matter situated in front of the section in question.

Divide horizontally the plane section or the back face of the sustaining wall into infinitely narrow bands. Any one of these bands situated at a certain distance L from the axis of z, that is from the top margin of the section, will have the height dL and its base or length equal to unity by hypothesis: it will suffer a total pressure $\mathfrak{R}dL$, composed of the normal force $-\mathfrak{N}dL$ and the tangential $\mathfrak{T}dL$ directed downward along a perpendicular in the section to its top boundary; $-\mathfrak{N}$, $\mathfrak{T}$, $\mathfrak{R}$ will have the values (72), (72^{bis}). The elementary pressures $\mathfrak{R}dL$ exerted on all of these bands will make with the normal to the section taken outward from the mass the constant angle ϕ_1 determined by the formula (72^{ter}), and they will be parallel; they will thus have a resultant or *total pressure* P equal to their sum and whose product to the distance L_1 of its point of application to the top boundary of the section will, after the theorem of moments, be equal to the sum of the respective products of the elementary pressures $\mathfrak{R}dL$ by the corresponding distances L of their points of application. This is expressed by the equations

$$(81) \qquad P = \int_0^L \mathfrak{R}dL, \qquad L_1 = \frac{1}{P}\int_0^L L\mathfrak{R}dL.$$

It only remains to effect the integrations indicated, after having substituted to $\mathfrak{N}$ its expression (72^{bis}) and put therein, instead of the normal distance l of each point of the section to the top slope, its value in function of the oblique distance L of the point in question to the same slope. This oblique distance L_1 estimated along the section, is inclined by ε_1 to the vertical, while the perpendicular l to the top slope is by ω : l is thus the projection of L under the angle $\omega-\varepsilon_1$ and have the value

$$(81^{bis}) \qquad l=L\cos(\omega-\varepsilon_1).$$

The formulae (81) become finally, by adding (72^{ter}),

$$(82)\begin{cases} L_1=\frac{2}{3}L, \quad P=K\frac{\rho g L^2}{2} \quad \text{where } K=\frac{\mathfrak{N}\cos(\omega-\varepsilon_1)}{\rho g l}=\frac{\sin\omega\cos(\omega-\varepsilon_1)\cos 2(\varepsilon_1-\varepsilon)}{\cos 2(\omega-\varepsilon)\sin\phi_1}, \\ tg(\phi_1+\varepsilon_1-\varepsilon\pm\frac{\pi}{4})=tg(\varepsilon_1-\varepsilon\pm\frac{\pi}{4})\frac{tg\frac{1}{2}(\frac{\pi}{2}-\omega+2\varepsilon\pm\omega)}{tg\frac{1}{2}(\frac{\pi}{2}-\omega+2\varepsilon\mp\omega)}. \end{cases}$$

A homogeneous pulverulent mass in elastic equilibrium terminated at the top by a plane slope with the inclination ω to the horizon, thus exerts, against a wall which has its back face inclined (in an internal batter) by an angle ε_1 to the vercital, a pressure applied at the third of the height of the back face of the wall inclined to the normal to this face (normal being taken inward to the wall) by the angle ϕ_1 given by the fourth equation (82), and equal to the half of the product of the weight of apparent unit volume of the mass by the square of the oblique height L of the face in question and by the coefficient K whose value results from the third formula (82). These laws are nevertheless demonstrated only for the modes of equilibrium in which the state of the matter is the same on all the extent of any plane parallel to the top slope and consequently depends on the sole parameter ε.

We have to consider the two components

$$\text{normal } P'=\int_0^L(-\mathfrak{N})dL \quad \text{and tangential } P''=\int_0^L \mathfrak{T}dL,$$

of the total pressure P. These two components will evidently have the expressions

$$P'=K'\frac{\rho g L^2}{2}, \quad P''=K''\frac{\rho g L^2}{2},$$

where the coefficients K', K'' denote the ratios $\frac{-\mathfrak{N}\cos(\omega-\varepsilon_1)}{\rho g l}, \frac{\mathfrak{T}\cos(\omega-\varepsilon_1)}{\rho g l}$,

that is, after (72), the numbers

$$K'=\frac{\cos(\omega-\varepsilon_1)}{\cos 2(\omega-\varepsilon)}[\cos(\omega-2\varepsilon)+\sin\omega\sin 2(\varepsilon_1-\varepsilon)]$$

$$K'' = \frac{cos(\omega-\varepsilon_1)}{cos2(\omega-\varepsilon)} sin\omega cos2(\varepsilon_1-\varepsilon) = K' tg(\omega-\varepsilon_1) + sin\varepsilon_1 :$$

the second expression of K'', namely $K'\ tg(\omega-\varepsilon_1)+sin\varepsilon_1$, is deduced immediately from the relation (72^{quater}).

36. *Values of the Pressure when the Wall, Rough or Smooth, is Fixed and the Natural State is Supposed to have Existed Beforehand.*

The third and the fourth of the formulae (82) are simplified when we admit, as special to the wall, the relations (73), that is when the pulverulent couch contiguous to the wall is supposed to be maintained in their position of natural state when it is rough and at least in its primitive plane when it is smooth. It must then be put, as we have seen at Arts. 22 and 23, $\varepsilon=\varepsilon_1$ in the first case, $\varepsilon=\varepsilon_1-\frac{\pi}{4}$ in the second. Also the value of ϕ_1 will be easily obtained from the two expressions (72) of $\mathfrak{T}$ and $-\mathfrak{N}$, whose ratio is equal to $tg\phi_1$. Thus

$$(82^{ter}) \qquad tg\phi_1 = \begin{cases} \dfrac{sin\omega}{cos(\omega-2\varepsilon_1)} & \text{(rough wall)}, \\ \text{zero} & \text{(smooth wall)}. \end{cases}$$

The third of (82), after the values (72^{bis}) of $\mathfrak{N}$, in the case of rough wall and observing that, in that of smooth wall, $\mathfrak{N}$ reduces to

$$-\mathfrak{N} = \frac{-sin(\omega-2\varepsilon_1)+sin\omega}{-sin2(\omega-\varepsilon_1)}\rho gl = \frac{2cos(\omega-\varepsilon_1)sin\varepsilon_1}{2cos(\omega-\varepsilon_1)sin(\varepsilon_1-\omega)}\rho gl = \frac{sin\varepsilon_1}{sin(\varepsilon_1-\omega)}\rho gl,$$

will give

$$(82^{quater}) \qquad K = \begin{cases} \dfrac{cos(\omega-\varepsilon_1)sin\omega}{cos2(\omega-\varepsilon_1)sin\phi_1} & \text{(rough wall)}, \\ \dfrac{sin\varepsilon_1}{tg(\varepsilon_1-\omega)} & \text{(smooth wall)}. \end{cases}$$

When the rough wall has its back face vertical or $\varepsilon_1=0$ and consequently $\phi_1=\omega$, it results simply

$$K = \frac{cos\omega}{cos2\omega}.$$

Observe lastly that, after the formula (82^{ter}), compared to the condition (69), the absolute value of $tg\phi_1$ will be always less than or at most equal to $sin\phi$; which signifies that the pressure supported by a sustaining wall undisturbing the pulverulent couch contiguous in its positions of natural state, makes with the normal to this couch an angle equal at most to that whose tangent equals the sine of the angle ϕ of internal friction. It will be thus sufficient, that such a wall may be supposed infinitely rough or capable of undisturbing the pulverulent couch contiguous in its primitive positions, that the angle of its friction against the mass shall be equal to or greater than that whose tangent equals the sine of the angle ϕ of internal friction (at $35^\circ\ 16'$ for $\phi=45^\circ$). We have always in the pratice, at the sustaining walls, enough of roughness that the angle of external friction shall be even greater than ϕ.