§ VI.

OF THE EQUILIBRIUM MODES WHICH CEASE TO BE POSSIBLE, WHENCE OF THE ELASTIC LIMIT OF THE PULVERULENT MATTER.

25. *Conditions Expressing that the Elastic Limits are not Exceeded.*

For all the bodies which tend to restore their form when we set them free, there exists what we call the *elastic limit*, that is the maximum values which the principal dilatations ∂_1, ∂_2, ∂_3 cannot surpass at each point, without that a persistent and sensible alteration of their initial molecular arrangement or even a rupture is produced. In particular, the pulverulent media, at the moment when their elastic limits are attained, present this state of unstable equilibrium which permits them to collapse and which we call *state of collapse* (*état ébouleux*).

The existence of an elastic limit in the pulverulent media is proved by this fact that they are destitute of cohesion, that is incapable to exert or transmit a tension however feeble. In other words, the normal component of the elastic force exerted on any plane element taken at their interior cannot be positive, and whence, after the first formula of (22) any of the three principal forces F_1, F_2, F_3, and even the greatest F_1, can never be positive. The mean $-p$ of these three forces then must be always negative, and the condition

$$F_1 < 0, \quad \text{or} \quad -p(1-2m\partial_1) < 0, \quad \text{or finally} \quad \partial_1 < \frac{1}{2m}$$

signifies that the greatest linear dilatation ∂_1 at the elastic state must always remain less than the ratio $\frac{1}{2m}$.

Let us confine ourselves to the case of plane deformations in which the linear dilatation ∂_2 is zero and which are, by virtue of the relation of incompressibility $\partial_1+\partial_3=0$ or $\partial_3=-\partial_1$, defined completely in their magnitudes, by means of the single positive dilation ∂_1, in a small extent about any point. The absence of cohesion of the medium shows ∂_1 to be less than the ratio $\frac{1}{2m}$, but not that ∂_1 can attain this limit without that the equilibrium is compromised All which we can conclude therefrom lies in this that the limit of elasticity is less than $\frac{1}{2m}$, or equal to $\frac{\sin\phi}{2m}$, if ϕ denote a certain angle, comprised between zero and $\frac{\pi}{2}$, which shall be determined by the experiments for each kind of pulverulent bodies and which shall be precisely what we call the *angle of internal friction.* Thus the two limiting conditions

$$(66) \qquad p>0, \qquad \partial_1 < \frac{sin\phi}{2m}$$

expressing the imperfection of elasticity of the pulverulent masses, shall be imposed to the equilibrium modes which these bodies can exhibit when we suppose them to be perfectly elastic, and they shall make the realization of those of the modes impossible which shall not be satisfied at all the points of the mass.

We can introduce, in the second inequality of (66), the extreme principal forces F_1, F_3 in place of the maximum dilatation ∂_1. The first equation (34^{quater}) [p. XXIX] gives $R=2mp\partial_1$, so that the inequality (66), $\partial_1 < \frac{sin\phi}{2m}$, becomes

$$(66^{bis}) \qquad \frac{R}{p}<sin\phi, \quad \text{or} \quad R<psin\phi.$$

Now after the relations (34^{ter}), we have

$$-\frac{F_1-F_3}{F_1+F_3}=\frac{R}{p},$$

and the inequality (66^{bis}) takes the required form

$$(66^{ter}) \qquad -\frac{F_1-F_3}{F_1+F_3}<sin\phi, \quad \text{or} \quad \frac{-F_1}{-F_3}>\frac{1-sin\phi}{1+sin\phi}=tg^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right).$$

Thus the ratio of the smallest stress to the greatest at the same point always exceeds $tg^2(\frac{\pi}{4}-\frac{\phi}{2})$; or, what comes to the same thing, the difference of these two stresses is less than the fraction $sin\phi$ of their sum.

The same inequality is yet susceptible to another interpretation. Consider a plane element whose normal is inclined by a certain angle β to the axis of x. The tangent of the angle made by the stress exerted on the plane element with the prolongation of this normal is equal to the ratio of the tangential component $\mathfrak{T}$ of this force to its normal component with its sign changed (*or the pressure properly called*) $-\mathfrak{N}$. Now the formulae (32) give

$$\frac{\mathfrak{T}}{-\mathfrak{N}}=\frac{R\,sin2(\beta-\beta_0)}{p+R\,cos2(\beta-\beta_0)},$$

and this ratio, which is zero when $sin2(\beta-\beta_0)=0$ or when $\beta-\beta_0=0$, $=\frac{\pi}{2}$, $=\frac{3\pi}{2}$, etc., attains its absolute maximum values when the derivative

$$\frac{d}{d\beta}\left(\frac{\mathfrak{T}}{-\mathfrak{N}}\right)=\frac{2R[p\,cos2(\beta-\beta_0)+R]}{[p+R\,cos2(\beta-\beta_0)]^2}$$

vanishes, that is when we have

$$(67) \qquad cos2(\beta-\beta_0)=-\frac{R}{p}.$$

Thus the inclination of the stress to the prolongation of the normal, the maximum inclination which I shall call ϕ', has for its tangent

$$tg\phi' = \frac{R\, sin2(\beta-\beta_0)}{p+R\, cos2\,(\beta-\beta_0)} = \frac{\pm\frac{R}{p}\sqrt{1-\frac{R^2}{p^2}}}{1-\frac{R^2}{p^2}} = \pm\frac{\frac{R}{p}}{\sqrt{1-\frac{R^2}{p^2}}};$$

this relation is equivalent to $sin^2\phi' = \frac{R^2}{p^2}$, and gives, after (66^{ter}),

$$(67^{bis}) \qquad sin^2\phi' < sin^2\phi.$$

Thus the second condition of (66) signifies also that *that the equilibrium may be possible the inclination of any pressure to the prolongation of the normal to the plane element of action, must always be less than or at most equal to the angle of internal friction ϕ.*

The inclination under question attains its absolute maximum value $\pm\phi'$ when we have

$$cos2(\beta-\beta_0) = -\frac{R}{p} = -sin\phi' = cos(\tfrac{\pi}{2}+\phi'),$$

and consequently, excepting an integral number of semicircumferences, when

$$\beta-\beta_0 = \pm\left(\frac{\pi}{4}+\frac{\phi'}{2}\right).$$

Now, after the reflections concluding Art. 15 (p. XXX), β_0 represents the inclination of the smallest principal force F_3 to the positive x: the excess $\beta-\beta_0$ then denote the angle between this force and the normal to the plane elements under question or that which the same plane elements make with the principal plane element subjected to the force F_3 under consideration. If lastly we observe that this force is negative and consequently constitute in its absolute value the greatest principal stresses, the above equality leads to the following theorem enunciated for the first time by Macquorn Rankine: *the plane elements for which the inclination of the stress thereby sustained to the prolongation of their normal attains its maximum value ϕ' are those which make an angle equal to $\frac{\pi}{2}+\frac{\phi'}{2}$ with the principal plane element subjected to the greatest stress, and which make in consequence with the plane element subjected to the least stress the complementary angle $\frac{\pi}{4}-\frac{\phi'}{2}$.*

26. *Characteristic Equation of the Equilibrium Limit of the Pulverulent Masses at the State of Collapse and of the Solids at the Plastic State.*

The inequality (66^{ter}) is changed to equality at the moment when the mass, at the point of its rupture, passes from the *elastic state* to the *state of collapse*: the *equilibrium limit* which is produced at this moment is then characterisea oy the relation

(68) $$F_1-F_3=-(F_1+F_3)\sin\phi.$$

The solid bodies also present an equilibrium limit when we apply to them sufficiently strong stresses; but the equation special to the *plastic state* which they produce can only be established by means of a little less simple considerations even when we confine ourselves to the study of a matter istropic that is constituted equally in all the directions, and to the case of the plane deformations such that one of the three principal dilations, ∂_2, is zero. These bodies, subjected to the unequal actions in different senses and very gradually increasing, begin to show the sensible permanent deformations as soon as the two other principal dilatations ∂_1, ∂_3 produced at a point acquire the values satisfying the relation $\partial_1-\partial_3=f(\partial_1+\partial_3)$, where f denotes a certain positive function: we then say that the *elastic limit* of the matter are *attained.* The deforming actions continuing to increase, the positions of *natural state* (or from which the elastic displacements are reckoned) of the varions particles which constitute the bodies change at each instant; and besides, the experiments show that if we make it resist to the disintegration by means of the suitably applied stresses, the body remains to be constituted with respect to these new positions of equilibrium (with the exception of a slight alteration of the isotropy) as it was in its initial state with respect to the first, its coefficients of elasticity λ, μ changing a little: but at the same time its molecular structure becomes more stable, since the function f, which measures at each instant the greatest possible elastic deformations $\partial_1-\partial_3$, is transformed and increases for a determinate value of the cubic dilatation $\partial_1+\partial_3$, such that the persistence of the same actions does not bring continuously new deformations. Lastly, the deforming actions still increasing there arrives a moment when the function f attains a maximum value which cannot be surpassed, and when, in consequence, the body is no more apt to assume a constitution which permits the deformations to be arrested. The equilibrium limit, characteristic of the plastic state, is produced at this moment. Now if in the relation $\partial_1-\partial_3=f(\partial_1+\partial_3)$ we substitute for ∂_1, ∂_3 their expressions deduced from the formulae (5) (in which we can take $A=0$, $\partial_2=0$), we have

$$F_1-F_3=2\mu f\left(\frac{F_1+F_3}{2\lambda+2\mu}\right),$$

which is the required equation.

In general, very malleable bodies, to which here we specially refer resist much less to the change of form than to that of volume, and the inverse, $\lambda+\frac{2}{3}\mu$ of their coefficient of compressibility must be very great with respect to their coefficient of rigidity μ; we can then suppose for the first approximation $2\lambda+2\mu=\infty$ when F_1+F_3 is not of a higher order than F_1-F_3 and the preceding equation is reduced to

$$F_1-F_3=2\mu f(0)=\text{a constant } 2K,$$

conforming to the fundamental principle of *plasticodynamics* which Mr. de Saint- Vénant has given as the result of the experiments of Mr. Tresca *Sur le poinçonnage des metaux.** But it is preferable to observe in addition the term of the first degree with respect to F_1+F_3 in the development of the function f by Maclaurin's series, which since f increases probably with the density or the mean pressure will give an equation of the form

$$(68^{bis}) \qquad F_1-F_3=2K-a(F_1+F_3).$$

We see that the formula of the state of collapse is nothing but a particular case of the above in which $K=0$ and $a=sin\phi$. In this particular case, the demonstration of it is more simple for two reasons: 1^0 for the pulverulent bodies, there are no different molecular constitutions which are susceptible to become in succession the more stable as the deforming actions increase, or in other words, the intermediate period during which the function increases, and which is improperly called the *period of the imperfect elasticity* does not exist in them; 2^0 the cubic dilatation $\partial_1+\partial_3$ is there sensibly equal to zero. Consequently, the equation of the equilibrium limit is reduced to $\partial_1-\partial_3=f(0)$ or $\partial_1=$ constant, as we have seen.

It is probable that the same causes of simplification occur for very malleable solid bodies as lead or clay, and the latter seems to require that we can suppose the maximum elastic value $\partial_1-\partial_3$ to depend little on $\partial_1+\partial_3$, or to admit approximately the equation of the equilibrium limit $F_1-F_3=$ a constant $2K$. This is also perhaps the case with the former, for which the function f, at the instant when the equilibrium is limiting, does not depend on the successive values assumed to this moment by the deformations ∂_1, ∂_3 or by the principal stresses F_1, F_3. If such is the case, the equation F_1-F_3 $=2\mu f\left(\frac{F_1+F_3}{2\lambda+2\mu}\right)$ will not perhaps apply, with a sufficient exactness, to the elastic bodies as iron which are suceptible to become considerably *springy* under the action of more and more considerable permanent loads, or for

* Refer in *Journal de Mathématiques* of M. Liouville (t. XVI, 1871) to various memoirs of Mr. de Saint-Venant on this subject, especially to that which is extracted from the *Compte-rendu* de la séance de 7 mars 1870 de l'Académie des science de Paris (t. LXX, p. 473) and in which the formula $F_1-F_3=2K$ is put in other forms. Very interesting memoir of Mr. Tresca on the punching has appeared at t. XX of the *Recueil des savants étrangers* of the same Academy 1872.

which the form of the function f, variable between large limits, will not always be the same at the instant when the plastic state is established.

27. *Application to the Equilibrium Modes Previously Considered.*

Now return, in view of the inequality (66), to the study of the mass with the indefinite length and depth and limited at the top by a plane inclined to the horizon by ω. The expression (60) of ∂_1, put in the second inequality of (66) raised to squares, is changed to

$$(69) \qquad cos^2(\omega-2\varepsilon) > \frac{sin^2\omega}{sin^2\phi}.$$

Two important consequences results from the fundamental relation (69):

1° The first member of this inequality being essentially less than unity, the second a fortiori must be the such, so that *the inclination ω of the slope to the horizon is always less than, or at most equal to, the angle ϕ of friction, in absolute value.* In fact, the experience has proved long since that the sand, the earth newly turned up, the heaps of small pebbles, etc., can only be sustained at angles less than a certain limiting angle, constant for the same kind of matter, but variable, for different kinds, from 24° or 26° (small lead shot, mustard seed) to 55° (most compact earth), and often taken equal to 45° in practice. The natural slopes of rupture as we observe for instance at the foot of steep rocks, along valleys or in mountains, are about 31° for fine and dry sand, 32° to 33° for marl, limestone and free earth thrown to wheelbarrow, 37° for chalky earth, 38° for moist quartzy sand, 45° for moist gypseous sand.*

2° The inequality (69) being satisfied at all the depths in the interior of the mass, the cosine of the angle $\omega-2\varepsilon$ cannot decrease in its absolute value to zero, or its tangent cannot increase indefinitely with l. After the formula (56), we must then have $c=0$, or $\varepsilon=$ constant; in other words, all the modes of equilibrium for which the constant c is not zero are irrealisable when the pulverulent medium is sufficiently deep. Consequently, *an indefinite mass does not permit the real modes of equilibrium different from those represented by equations* (57) *and* (58), *when we suppose the angular parameter ε to be constant, or when the deformations suffered are equal at all the points.*

Nothing needs to be engrossed with the first inequality of (66). In

* *Mémoires sur la poussée des terres*, by Mr. Saint-Guilhem, ingenieur en chef des ponts et chaussées (*Annales des ponts et chaussées*, t. XV, May and June 1858), note VII.—Also see the *Mécanique appliquée* of Mr. Collignon, t. I, p. 478.

fact, after (56), the expression (50) of p is nothing but

$$p = \frac{\rho g l}{\cos\omega - \sin\omega \, tg\,(\omega - 2\varepsilon)},$$

and it will be greater than zero when the positive term $\cos\omega$ of the denominator is in absolute value greater than $\sin\omega \; tg(\omega - 2\varepsilon)$. Now the inequality (69) can be written

$$\frac{\sin^2\omega}{\sin^2\phi} < \frac{1}{1 + tg^2(\omega - 2\varepsilon)};$$

whence we deduce

$$\sin^2\omega \; tg^2(\omega - 2\varepsilon) < \sin^2\phi - \sin^2\omega,$$

and a fortiori,

$$\sin^2\omega \; tg^2(\omega - 2\varepsilon) < 1 - \sin^2\omega = \cos^2\omega,$$

so that the absolute value of $\sin\omega \; tg(\omega - 2\varepsilon)$ is much less than $\cos\omega$. The inequality (69), in the study of the equilibrium modes of the indefinite mass; thus holds the place, in itself alone, of the two inequalities (66) which we had to consider.

28. *Limits between which the Angular Parameter* ε, *measuring the Inclination to the Vertical of the Non-dilating and Non-contracting Couches, can be Varied.*

Now continue the study of the inequality (69).

Let τ be the angle comprised between zero and $\frac{\pi}{2}$, whose cosine has the value

$$(70) \qquad \cos\tau = \frac{\sin\omega}{\sin\phi} \text{ (in absolute value)}.$$

As the equilibrium mode corresponding to a certain value of ε subsists without the modification when ε increases by $\frac{\pi}{2}$, we can confine ourselves to consider, whatever ω may be, the values of ε comprised in an interval equal to $\frac{\pi}{2}$, and suppose in consequence $\omega - 2\varepsilon$ to be variable from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ only. Thus we shall have $\cos(\omega - 2\varepsilon) > 0$, and the inequality (69) becoming

$$\cos(\omega - 2\varepsilon) > \cos\tau,$$

will be equivalent to the ensemble of the two:

$$(70^{bis}) \qquad \omega - 2\varepsilon \begin{cases} > -\tau, \\ < \tau, \end{cases} \quad \text{or} \quad \varepsilon \begin{cases} > \dfrac{\omega - \tau}{2}, \\ < \dfrac{\omega + \tau}{2}. \end{cases}$$

Given any value of ω of the inclination of the slope to the horizon, the equilibrium is then possible: 1° when the inclination ε to the vertical of a non-dilating and non-contracting couch (or consequently of the rough back face of a sustaining wall which will not move such a couch in their positions of natural state) is exactly equal to $\frac{\omega}{2}$; 2° when this inclination is less than or greater than $\frac{\omega}{2}$ by a quantity at most equal to the half of the angle τ defined by equation (70), or, in other words, when the direction of a couch of invariable extent is comprised in the angle τ so constructed that its bisector may be inclined at $\frac{\omega}{2}$ to the vertical; 3° when the inclination ε differs from one of the preceding by only an integral number of right angles that is, in general, when the direction of the couch under consideration is comprised in the interior of one of the four angles, equal to τ and opposite in pairs at the vertex, which have for respective bisectors the straight line inclined by $\frac{\omega}{2}$ to the vertical and its perpendicular.

On the contrary, the equilibrium becomes impossible, with the given value of the inclination ω of the slope, when the direction of a couch of invariable extent (or of the rough back face of a sustaining wall which will not move it in its positions of natural state) falls in the interval left out between these four angles. Finally the preceding inequalities change to equalities and the elastic limit is precisely attained, when ε acquires its extreme values, $\frac{\omega \mp \tau}{2}$ plus an integral number of right angles, which takes place when the invariable couches are found exactly parallel to a side of these four angles; the state then becomes *collapsing* and the equilibrium *limiting*.

For $\omega=0$, the angle τ is a right angle and consequently the four angles under question comprise all the space around the point O, or do not permit the existence of any direction ε for which the equilibrium may be impossible. But, as the absolute value of ω increases, τ decreases more and more until it vanishes when ω attains its two extreme values $\pm\phi$. This decrease of τ is also more rapid than the increase of $\pm\omega$; for the differentiation of (70) gives

$$\frac{-d\tau}{d\omega}=\frac{\cos\omega}{\sin\phi\sin^2\tau}=\frac{\cos\omega}{\sqrt{\sin^2\phi-\sin^2\omega}}=\sqrt{\frac{1-\sin^2\omega}{\sin^2\phi-\sin^2\omega}}>1 \text{ (in absolute value).}$$

Consequently, *when ω increases from zero to ϕ or decreases from zero to $-\phi$, the two extreme values of ε, $\frac{\omega-\tau}{2}$ and $\frac{\omega+\tau}{2}$, initially equal to $-\frac{\pi}{4}$ and $\frac{\pi}{4}$ respectively, vary continuously each in a same sense, the first in augmenting and the second in diminishing up to their common final value which is the half of that of ω, i. e., $\pm\frac{\phi}{2}$. The four conjugate angles τ which form, by their symmetrical grouping around the point o, a kind of Maltese cross whose arms they occupy, and in the interior of which is found at each instant the admissible directions of the invariable couches, contract all together, for a simultaneous movement of retreat of their four sides in such a manner as to leave* *out between them a greater and greater number of directions which cease to be admissible for all the subsequent values of ω.**

29. *Limits between which the Inclination ω of the Slope must be Comprised for a Given Value of ε.*

The two limits, one negative and the other positive, between which the inclination ω to the horizon of the top slope must be comprised that the equilibrium may be possible for a given inclination ε of the invariable couches to the vertical are easily obtained. The inequality (69) is changed to equality when ω acquires these two values, and extracting the square root of the two members, it gives

$$sin\omega = \pm sin\phi cos(\omega - 2\varepsilon) = \pm sin\phi (cos\omega\ cos2\varepsilon + sin\omega\ sin2\varepsilon),$$

or, solving with respect to $tg\omega$,

$$(71) \qquad tg\omega = \frac{\pm sin\phi\ cos2\varepsilon}{1 \mp sin\phi\ sin2\varepsilon} = \frac{\pm cos2\varepsilon}{\frac{1}{sin\ \phi} \mp sin2\varepsilon}.$$

To these two values of $tg\omega$ correspond respectively the required two limiting angles, one comprised between zero and $+\phi$ and the other between zero and $-\phi$. We observe that they simply change their signs when ε is changed to $-\varepsilon$, and we can confine ourselves to calculate them for the positive values of ε. It is also sufficient to make ε vary from zero to $\frac{\pi}{4}$ or 45°, since the same circumstances are produced for all the values of ε which differ from one another by an integral number of right angles.

If we adopt for ϕ the value $\frac{\pi}{4}$ or 45°, the expression of $tg\omega$ becomes

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* When the mass is solid, δ_1, δ_3 have the values (60^{bis})(p. XXXIX). We see that these principal dilatations, if they are not negative, will put an end by surpassing every admissible elastic limit *in a mass sufficiently deep* when we shall take l or p great enough. We shall have then $\delta < 0$; or, by putting

$$(a) \qquad \frac{\mu}{\lambda + \mu} = sin\phi,$$

where ϕ denotes an acute angle, we return to the first inequality (66) and the inequality (69), that is precisely to the two same conditions as for a pulverulent mass, resumed in the unique condition (69). Thus, all those which have been said in this § on the subject of the limits between which ω and ε are comprised apply to a solid mass ef great depth, except that the equilibrium will be stable for the extreme values $\pm\tau$ of $\omega - 2\varepsilon$ (observe that we have then $\delta_1 = 0$, $\delta_3 < 0$). In particular, *the inclination ω of the slope must be less than ϕ*, as Mr. de Saint-Venant had already recognised. This angle ϕ will be 30° in *hard* bodies in which it is probable that the ratio $\frac{\mu}{\lambda}$ does not differ sensibly from unity, and it will be nearly zero for *soft* bodies, in which the ratio $\frac{\mu}{\lambda}$ is doubtless near to zero.

$$tg\omega = \frac{\pm cos 2\varepsilon}{\sqrt{2} \mp sin 2\varepsilon},$$

and we can form the following table:

Values of ε	0,	10°,	20°,	30°,	40°,	45°,
limiting values of ω	35°16′,*	41°14′,	44°48′,	42°22′,	22°01′,	0,
	−35°16′,	−28°09′,	−20°26′,	−12°22′,	−4°08′,	0.

By differentiating equation (71), we find

$$\frac{d\,tg\omega}{d\varepsilon} = \frac{\pm 2 sin\phi(-sin 2\varepsilon \pm sin\phi)}{(1 \mp sin\phi sin 2\varepsilon)^2}.$$

When ε increases from zero to $\frac{\pi}{4}$, the negative value of ω that is the lower limit, increases continuously to approach to zero. As to the upper limit, or extreme positive value of ω, it increases at first up to a maximum ϕ which it attains for $\varepsilon = \frac{\phi}{2}$ (that is for $\varepsilon = 22°\frac{1}{2}$ when $\phi = 45°$), and it decreases afterwards. The two limits, initially equal in absolute magnitudes to arc tg(sinϕ) but of opposite signs, vanish in latter case for $\varepsilon = \frac{\pi}{4}$.

We can, by means of the tangents of these limiting angles which I will call ω', ω'',

$$tg\omega' = \frac{sin\phi cos 2\varepsilon}{1 - sin\phi sin 2\varepsilon}, \quad tg\omega'' = \frac{-sin\phi cos 2\varepsilon}{1 - sin\phi sin 2\varepsilon},$$

calculate the tangent of their difference $\omega' - \omega''$. It reduces to, after some evident simplifications,

$$tg(\omega' - \omega'') = \frac{2tg\phi}{cos\phi} cos 2\varepsilon:$$

it diminishes continuously when ε increases from zero to $\frac{\pi}{4}$. *The deviation $\omega' - \omega''$ of the two limits is then the greater the less the inclination ε or the less the invariable couches deviate from the vertical.*

We observe that a rough sustaining wall, inclined by ε to the vertical and which will be supposed not to move the contiguous pulverulent couch in its positions of natural state, could not sustain a mass limited at the top by a plane slope whose inclination to the horizon will be either greater than the positive limit ω' or less than the negative limit ω''. No doubt the part BB' (fig. 3) of such as lope which will be very far from the sustaining wall could receive any declivity less than or even equal, in absolute value, to the angle of friction ϕ. But the surface of the slope must then cease to be plane as we approach near the wall OM, so as to become concave toward the vertical for positive ω, convex in the contrary case.

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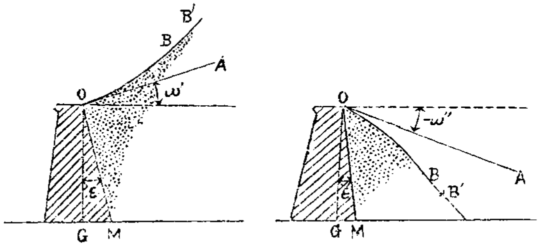
* For fine dry sand we shall have only $\phi = 31°$, and the limiting values which correspond to $\varepsilon = 0$ will become $\pm 27°\ 15'$.

and no more, very near to the wall, to have its inclination ω to the horizon



(Fig. 3)

on outside of the two limits ω', ω'' The formulae above established are not applicable to such curved slopes, and the rigorous calculation of the equilibrium which will be then produced is perhaps inaccessible; but if the curveture of the surface were little sensible and if we propose to know the circumstances produced at sufficiently small distances only from the origin O, we shall commit merely a negligible error by replacing the true profil of the slope by its tangent along OA, the tangent whose inclination to the horizon will be equal to the corresponding limiting value ω' or ω''. In other words, the case where the angle ω is found to be greater than ω' could be approximatly confounded with that is which $\omega=\omega'$, and the case where ω will be less than ω'' could similarly be not distingnished from the case $\omega=\omega''$.

Besides, this remark will not appear to have practical importance, and for two reasons. The first consists in this that the rough sustaining wall which we construct shall not undisturb the contiguous pulverulent particles in the positions of natural state, or nearly in that which concern to the couches of earth or of sand not yet compressed during the period itself of formation of the mass: as we bring new couches on the preceding the latter suffers a large number of ruptures, in consequence of which the positions of natural state of their cles are found entirely changed, as we shall see at § VIII. The second reason consists in this that the inclination ε tothe vertical of the rough sustaining wall, supposed also to be capable to undisturb the contiguous earth particles in their positions of natural state, has, almost always in practice, such values that the upper limit ω' is near to its maximum ϕ, while the negative limit ω'' is found much below the inclinations of the slopes which we can employ.

Nevertheless, while we discharge against the back face of a wall the earth or the sand which it must sustain, the negative values of ω less than ω'' are often produced, and it appears then that we verify the convex form of the slope indicated by the preceding theory.

30. *Limits which contain ω for a Given Value of ε'.*

I have above considered only a wall with plane and rough back face. Suppose now this face to be infinitely smooth and inclined to the vertical by an angle ε', but always so given as to prevent the contiguous superficial couch of the mass from leaving its initial plane; we have seen at Art. 23 (p. XLV) that the equilibrium in these conditions does not differ from that will be produced by a rough wall inclined by $\varepsilon=\varepsilon'-\frac{\pi}{4}$, or by $\varepsilon=\varepsilon'+\frac{\pi}{4}$, inasmuch as a difference of $\frac{\pi}{2}$ in these angles causes no change in the mode of equilibrium. We shall have then, in particular, the two limits ω' ω'' which contain between them all the admissible values of the inclination of the slope by extending, by means of an observation which follows the formula (71), the preceding table to the negative values of ε varying from $-\frac{\pi}{4}$ to zero and then adding for ε' $\frac{\pi}{4}$ or 45° to all the values of ε registered in the table. We have thus

for $\varepsilon'=$	0,	5°,	15°,	25°,	35°,	45°,	55°,	65°,	75°,	85°,	90°,
limiting values of ω	0,	4°08′,	12°22′,	20°26′,	28°09′,	35°16′,	41°14′,	44°48′	42°22′,	22°01′,	0,
	0,	-22°01′,	-42°22′,	-44°48′,	-41°14′,	-35°16′,	-28°09′,	-20°26′,	-12°22′,	-4°08′,	0.

The four angles, equal to τ and symmetrically disposed in a cross around the origin O, which contain, for a given value of ω, all the admissible directions of the back face of the wall, remain the same in magnitude as in the case of a rough wall; but they are turned by half a right angle so as to be precisely in the middle of the four gaps left between them. If the angle τ is greater than half a right angle there will be, in the midst of each of the four branches of a sort of two crosses, an angular space, equal to $\frac{\pi}{4}-\tau$, which it has not in common with the other cross; but there will be, on each margin of the branch in question, a band equal to $\tau-\frac{\pi}{4}$ and common to the two crosses. Along each direction comprised in one of these eight bands of angular magnitude $\tau-\frac{\pi}{4}$ and symmetrically disposed with respect to the bisectors of the eight angles formed between the axes of the two crosses, we can take with the exclusion of the rest of the plane, the back face of a sustaining wall, whether rough or smooth, capable of maintaining the equilibrium under the hypothesis to which I have been subjected with the special relations (37). The necessary and sufficient condition that these bands may exist consists in that τ shall be greater than $\frac{\pi}{4}$ or $\cos\tau<\frac{1}{\sqrt{2}}$, that is, after (70), that we shall have

$$\sin\omega<\frac{\sin\phi}{\sqrt{2}}$$

in absolute value, or, by supposing $\phi=45^\circ$,

$$\sin\omega<\tfrac{1}{2},\qquad \omega<30^\circ.$$