§ IV.

THEIR INTEGRATION WHEN THE MASS IS BOUNDED AT THE TOP BY A PLANE AND INFINITE IN OTHER DIRECTIONS.

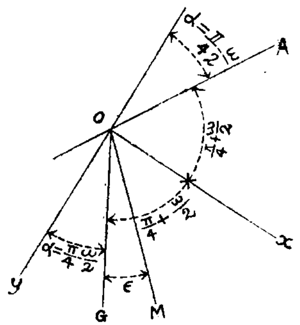
17. *First Integration.*

Consider at first a mass bounded at the top by a plane and infinite in other senses, or, what comes to the same thing, comprised laterally between two infinitely smooth walls perpendicular to a horizon at the top slope. By virtue of symmetry, the displacements will take place in the planes normal to this horizon and in the same manner in all of them. If, then, we take one of these vertical planes for that of xy, the preceding formulae established for the case of plane deformations can be employed.

Let OA (fig. 1) be a line of greatest inclination of the free surface or of the top slope in the initial state of the mass and OG a vertical drawn downward. I will take for the axis of x the bisector of the angle GOA and for that of y a perpendicular to Ox in such a manner that the angle GOy may be acute. If we denote by ω the initial inclination of the slope on the horizon (inclination variable at most from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$), the quantity α which denotes, in the formulae (28), Art. 14, the angle made by the vertical with the axis of y will evidently be $\frac{\pi}{4}-\frac{\omega}{2}$, and we shall have at once

(38) $$GOA=\frac{\pi}{2}+\omega,\quad \alpha \text{ or } GOy=\frac{\pi}{4}-\frac{\omega}{2},$$

The mass being indefinitely parallel to OA, all the particles of matter situated at the same distance from the surface are exactly in the same



(Fig. 1.)

conditions. In other words, the quantities N_1, N_2 T and cousequently p, ∂_x, ∂_y g_{xy} at each point are fnnctions of only the initial distance l of the point under consideration to the surface. The perpendicular l drawn from any point (xy) of the mass to meet with OA makes the angles α and $\frac{\pi}{2}-\alpha$ with the axes of x and y respectively, and we have

$$(39) \qquad l = x\cos\alpha + y\sin\alpha,$$

In consequence, the derivatives in x and y of every function of l will be obtained from its derivatives in l by the symbolic formulae

$$(40) \qquad \frac{d}{dx} = \cos\alpha\,\frac{d}{dl}, \qquad \frac{d}{dy} = \sin\alpha\,\frac{d}{dl},$$

and the indefinite equations (28) can be written

$$(41.) \qquad \begin{cases} \dfrac{d}{dl}\left[N_1\cos\alpha + (T+\rho gl)\sin\alpha\right] = 0, \\[2ex] \dfrac{d}{dl}\left[(T+\rho gl)\cos\alpha + N_2\sin\alpha\right] = 0, \end{cases}$$

The normal to the free surface OA making the angle α with the axis of x, the special conditions (35), in which γ is to be replaced by α and which must be satisfied at this surface, show that the expressions

$$N_1\cos\alpha + (T+\rho gl)\sin\alpha, \qquad (T+\rho gl)\cos\alpha + N_2\sin\alpha$$

vanish for $l=0$. These expressions, after, (41), should then vanish everywhere, and we shall have

$$(42) \qquad \begin{cases} N_1\cos\alpha + (T+\rho gl)\sin\alpha = 0, \\ (T+\rho gl)\cos\alpha + N_2\sin\alpha = 0, \end{cases}$$

By adding the two fundamental equations (42) after having multiplied them respectively by $\cos\alpha$, $-\sin\alpha$ and by $\sin\alpha$, $\cos\alpha$, and by substituting in the results the values

$$\tfrac{1}{2}(1+\cos2\alpha), \quad \tfrac{1}{2}(1-\cos2\alpha), \quad \tfrac{1}{2}\sin2\alpha, \quad -p,$$

or, after (38),

$$\tfrac{1}{2}(1+\sin\omega), \quad \tfrac{1}{2}(1-\sin\omega), \quad \tfrac{1}{2}\cos\omega, \quad -p.$$

for $\cos^2\alpha$, $\sin^2\alpha$, $\cos\alpha\sin\alpha$, $\frac{1}{2}(N_1+N_2)$, we find that they come to the two following equations

$$(43) \qquad \begin{cases} \tfrac{1}{2}(N_1-N_2) - p\sin\omega = 0, \\ T+\rho gl - p\cos\omega = 0. \end{cases}$$

18. *Second Integration*

Replace, in the first equation of (43), N_1 and N_2 by their values (27), in which the two dilatations ∂_x, ∂_y are equal and of opposite signs. Since the mean stress p is evidently not zero in the interior of the mass, we shall have

$$\partial_x - \partial_y = \frac{\sin\omega}{m},$$

i. e.,

(44) $$\eth_x \text{ or } \frac{du}{dx}, = \frac{sin\omega}{2m}, \quad \eth_y \text{ or } \frac{dv}{dy} = -\frac{sin\omega}{2m}.$$

These equations (44) with the introduction of two arbitrary functions ϕ and ψ give by their integrations

(45) $$u = \frac{sin\,\omega}{2m}\,[x + \phi(y)], \qquad v = \frac{sin\omega}{2m}[-y + \psi(x)].$$

We deduce therefrom

(46) $$g_{xy} \text{ or } \frac{du}{dy} + \frac{dv}{dx} = \frac{sin\omega}{2m}[\phi'(y) + \psi'(x)].$$

Now the deformation g_{xy} must depend only upon the distance l of the point under consideration to the free surface as already remarked, and the symbolic formulae (40) which are consequently applicable to them show that its two derivatives in x and y must be in the ratio of $cosa$ to $sina$, so that

(47) $$\frac{\psi''(x)}{cosa} = \frac{\phi''(y)}{sina},$$

and the two ratios (47), the first independent of y and the second of x, can only reduce to a same constant $2c$. Two successive integrations consequently give, whith the introduction of four arbitrary constants c', c'', c'_1, c''_1,

$$\phi(y) = cy^2 sina + (c' + c'')y + c'_1,$$
$$\psi(x) = cx^2 cosa + (c' - c'')x + c''_1.$$

The expressions (45) of u and v thus become

(48) $$\begin{cases} u = \dfrac{sin\omega}{2m}[x + cy^2 sina + (c' + c'')y + c'_1], \\ \\ v = \dfrac{sin\omega}{2m}[-y + cx^2 cosa + (c' - c'')x + c_1''], \end{cases}$$

and those, (44), (46), (27), of $\eth_x$, $-\eth_y$, g_{xy}, N_1, N_2, T are in their order:

(49) $$\begin{cases} \eth_x = -\eth_y = \dfrac{sin\omega}{2m}, \qquad g_{xy} = \dfrac{sin\omega}{m}(c' + cl), \\ \\ N_1 = -p(1 - sin\omega), \quad N_2 = -p(1 + sin\omega), \quad T = p(c' + cl)sin\omega. \end{cases}$$

As to the mean stress p, it results from the second equation of (43) in which we shall replace T by its value (49). The solution of this equation with respect to p gives in consequence

(50) $$p = \frac{\rho g l}{cos\omega - (c' + cl)sin\omega}$$

19. *Transformed Lines of a Family of Parallel Material Straight Lines.*

Of five arbitrary constants c, c', c'', c'_1, c''_1, the first two only enter in the expressions of the deformations $\eth$, g encountered by the mass. The three others in fact correspond to a small motion only, viz., c'' to a small rotation of the mass around the origin O, and c'_1, c_1'' to two translations parallel to the axes; now this motion remains indeterminate so long as the mass is supposed to be infinite and consequently its relations with the fixed bodies which touch it at a more or less distance from the origin of the coordinates are disregarded.

We can thus suppose $c''=0$, $c'_1=0$, $c''_1=0$ when we have only to study the form taken by any material line whose equation was $f(x,y)=0$ before the displacement. If x', y' denote, in the new state of equilibrium, the coordinates of the material point which was initially at (x,y), we shall have, after (48),

$$(51)\qquad \begin{cases} x=x'-u=x'-\dfrac{sin\omega}{2m}(x'+cy'^2sina+c'y') \text{ nearly,} \\ y=y'-v=y'-\dfrac{sin\omega}{2m}(-y'+cx'^2cosa+c'x') \text{ nearly,} \end{cases}$$

and the transformed curve of $f(x, y)=0$ will be

$$(52)\quad f[x'-\frac{sin\omega}{2m}(x'+cy'^2sina+c'y'),\ y'-\frac{sin\omega}{2m}(-y'+cx'^2cosa+c'x')]=0.$$

Every family of parallel straight lines

$$(53)\qquad xcosA+ysinA=k,$$

where A denotes the constant inclination of their normals to the axis of x, and k, the parameter variable from one straight line to the other, their distance to the origin, transforms into a family of conics having for equation

$$(54)\quad x'[(1-\frac{sin\omega}{2m}cosA-\frac{c'sin\omega}{2m}sinA]+y'[(1+\frac{sin\omega}{2m})sinA-\frac{c'sin\omega}{2m}cosA]$$
$$-\frac{c\,sin\omega}{2m}(y'^2sina\,cosA+x'^2cosasinA)=k,$$

or rather

$$(55)\quad cosa\,sinA\left[x'-\frac{\left(\frac{2m}{sin\omega}1-\right)cosA-c'sinA}{2c\,cosa\,sinA}\right]^2$$
$$+sina\,cosA\left[y'-\frac{\left(\frac{2m}{sin\omega}+1\right)sinA-c'cosA}{2c\,sina\,cosA}\right]^2$$
$$=\frac{\left[\left(\frac{2m}{sin\omega}-1\right)cosA-c'sinA\right]^2}{4c^2cosa\,sinA}+\frac{\left[\left(\frac{2m}{sin\omega}+1\right)sinA-c'cosA\right]^2}{4c^2sina\,cosA}-\frac{2mk}{c\,sin\omega}.$$

These conics are similar, concentric and have their axes parallel to those of x and y, that is to the two bisectors of the four angles formed by a vertical and the profile of the top slope. They are reduced to circles when tne proposed straight lines are parallel to the free surface of the mass or when $A=\alpha$, and to parallel straight lines, after (54), in the case where the constant c is zero.

The latter result would be directly deduced from the values (49) of ∂_x, $-\partial_y$, g_{xy} which become constant when $c=0$: the material plane elements initially rectangular, cut up in the plane of xy by a double infinity of equidistant straight lines parallel to the coordinate axes, are there changed, by the deformations produced, to equal parallelograms, and the trains of points thus formed which were initially situated along two parallel straight lines do not discontinue to be disposed in two rectilinear rows both having the same orientation.

20. *Elastic Forces Parallel to the Plane of Deformations; Correlative Dilatations and Shears.*

To find the two components, normal $-\mathfrak{N}$ and tangential $\mathfrak{T}$, of the stress (elastic force with its sign changed) exercised on the plane element parallel to the axis of z, which makes a certain angle ε_1 with the vertical or whose normal makes the same angle ε_1 with the horizon, the angle $\omega-\varepsilon_1$ with the top slope OA and finally the angle $-(\frac{\pi}{4}-\frac{\omega}{2}+\varepsilon)$ with positive x, it will be sufficient to put in the formulae (30) the values (49), (50) of N_1, N_2, T, p and then to make $\beta=-(\frac{\pi}{4}-\frac{\omega}{2}+\varepsilon_1)$ or $2\beta=-\frac{\pi}{2}+(\omega-2\varepsilon_1)$. Thus we find

$$-\mathfrak{N}=p\{1+sin\omega[(c'+cl)cos(\omega-2\varepsilon_1)-sin(\omega-2\varepsilon_1)]\},$$
$$\mathfrak{T}=p\ sin\omega[(c'+cl)sin(\omega-2\varepsilon_1)+cos(\omega-2\varepsilon_1)].$$

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(56) $$c'+cl=tg(\omega-2\varepsilon),$$

ε denoting an auxiliary angle GOM, *which becomes sensibly constant for very large l, whatever c may be*: we choose its value in general such that the difference $\omega-2\varepsilon$ may be comprised between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, although it can have yet this value increased or diminished by any multiple of $\frac{\pi}{2}$. The above expressions of $-\mathfrak{N}$, $\mathfrak{T}$ and that (50) of the mean stress p then become:

(57) $$\left\{\begin{aligned} p&=\frac{\rho gl cos(\omega-2\varepsilon)}{cos2(\omega-\varepsilon)},\\ -\mathfrak{N}&=\frac{\rho gl}{cos2(\omega-\varepsilon)}[cos(\omega-2\varepsilon)+sin\omega\ sin2(\varepsilon_1-\varepsilon)],\\ \mathfrak{T}&=\frac{\rho gl}{cos2(\omega-\varepsilon)}sin\omega\ cos2(\varepsilon_1-\varepsilon).\end{aligned}\right.$$

It is easy thereby to deduce: 1° the value $\partial_{x'}$ of the small dilatation suffered by a material line initially normal to the plane element or inclined above the horizon by the angle ε_1; 2° the small cosine $g_{x'y'}$ of the angle made by this material line, after the displacements, with that which was normal to it and whose inclination to the vertical (directed downward) was equal to ε_1. In fact, these values are deduced from the general expressions of the elastic forces parallel to the plane of deformations

$$-\mathfrak{N}=p(1-2m\partial_{x'}), \quad \mathfrak{T}=pmg_{x'y'},$$

when we substitute for $-\mathfrak{N}$, $\mathfrak{T}$, p their expressions (57). It will then become

$$(58) \qquad \partial_{x'}=-\frac{\sin\omega \sin 2(\varepsilon_1-\varepsilon)}{2m\cos(\omega-2\varepsilon)}, \quad g_{x'y'}=\frac{\sin\omega\cos 2(\varepsilon_1-\varepsilon)}{m\cos(\omega-2e)}.$$

The formulae (56), (57), (58) will be of a great use to us. I shall now be content with deducing from the last two some consequences almost evident.

The dilatation $\partial_{x'}$ is zero when $\varepsilon_1=\varepsilon$, i.e., for the rectilinear material element which is inclined above the horizon by the angle ε defined by equation (56), and it is also zero for the rectilinear element, initially perpendicular to the former, which makes the same angle ε with the vertical or the angle $\varepsilon-\frac{\pi}{2}$ with the horizon, the angle which can equally be taken for value of ε satisfying equation (56) without that the equilibrium mode is thereby in any way modified. These two linear elements and their opposite are, at each point, the only one which do not suffer any variation of magnitude; one of them rotates with respect to the other so as to reduce their angle by the small quantity $g_{x'y'}=\frac{\sin\omega}{m\cos(\omega-2\varepsilon)}$, which just acquires its maximum absolute value in these two straight lines.

It is along the bisectors of the four angles formed by the directions thus defined that the two principal linear dilatations ∂_1, ∂_3 are produced. I will denote by ε' the inclination above the horizon of any one of these intermediate directions, so that

$$(59) \qquad \varepsilon'=\varepsilon\mp\frac{\pi}{4},$$

denoting by ε, as is seen to be permissible, the inclination above the horizon of one of the rectilinear elements neither contracted nor elongated and making an angle of 45^0 with the principal dilatation under consideration. These principal dilatatious, one of which ∂_1 is positive and the other ∂_3 negative, have the values, after the first formula of (58),

$$(60) \qquad \partial_1=\pm\frac{\sin\omega}{2m\cos(\omega-2\varepsilon)}, \quad \partial_3=\mp\frac{\sin\omega}{2m\cos(\omega-2\varepsilon)}.$$

The rectilinear material elements which suffer these dilatations rem in always rectangular, since the second formula of (58) gives $g_{x'y'}=0$ when we make therein $\varepsilon_1=\varepsilon'=\varepsilon\mp\frac{\pi}{4}$.

We shall see later on that the only modes of equilibrium which are useful for our considerations are those for which the arbitrary constant c is zero, making thereby, after (56), the value of $tg(\omega-2\varepsilon)$ and that of ε constant. *Thus a double system of respectively equidistant material lines, one initially inclined to the vertical and the other to the horizon by the angle* ε *and dividing a normal section of the mass in equal squares, are yet rectilinear, parallel and have the same lengths after the deformations: although they are rotated from each other by the small angle* $\frac{\sin\omega}{m\cos(\omega-2\varepsilon)}$, *and the squares comprised by them are thus transformed into equal lozanges. Consequently, the definitive form of the mass is simply obtained if we conceive it to be divided into infinitely thin couches in the initial state all inclined by the acute angle* ε *to the vertical, and then we make them slide in their respective planes, such that, one of them remaining fixed, every other situated at a distance D in front of the first couch displaces (downward) by the quantity* $\frac{D\sin\omega}{m\cos(\omega-2\varepsilon)}$.*

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* For a mass solid and not pulverulent, the formulae (38) to (43) always give

$$(49^{bis})\qquad N_1=-p(1-\sin\omega),\ N_2=-p(1+\sin\omega),\ T=p\cos\omega-\rho gl,$$

and, p depending on l only, the relation (28^{ter}) (p. XXVII), reduced to

$$\frac{\lambda+2\mu}{\lambda+\mu}\frac{d^2p}{dl^2}=0$$

shows that p *is a linear function of* l. The formulae (57) do not discontinue to be applicable, if we continues to take

$$(56^{bis})\qquad tg(\omega-2\varepsilon)=\frac{1}{\sin\omega}\left(\cos\omega-\frac{\rho gl}{p}\right):$$

ε *thus becoming yet sensibly constant at sufficiently great depths* l. We easily find by means of the formulae (24) (where $A=0$) and (57) that the two principal dilatations ∂_1, ∂_3 have then the values:

$$(60^{bis})\qquad \partial=-\frac{p}{2\mu}\left[\frac{\mu}{\lambda+\mu}\pm\frac{\sin\omega}{\cos(\omega-2\varepsilon)}\right].$$

Observe that, after the relation (56^{bis}) which is also restored to the first equation of (57) the angle $\omega-2\varepsilon$ varies continuously in the same sense when the ratio $\frac{p}{\rho gl}$ increases continuously from $-\infty$ to ∞; this angle ε thus changes from $-\frac{\pi}{2}-\omega$ to $\frac{\pi}{2}-\omega$ if ω is positive and from $\frac{\pi}{2}-\omega$ to $-\frac{\pi}{2}-\omega$ if ω is negative, its absolute value being greater than, equal to or less than $\frac{\pi}{2}$ according as the ratio $\frac{p}{\rho gl}$ is negative, zero or positive.

§ V.

STUDY OF THE SAME MASS WHEN WE SUPPOSE IT NO MORE TO BE INFINITE BUT RETAINED AT ONE SIDE BY A PLANE WALL WHICH INTERSECTS ITS TOP SLOPE ALONG A HORIZON.

21. *The Formulae Obtained For an Infinite Mass are Somtimes Applicable to the Limited Masses.*

Of all the mode of equilibrium represented by the formulae (48), (49), (50) the most interesting one are those in which they are satisfied for the whole length of a line situated in the plane of xy, the conditions at a wall being, for instance, the first two or the last two of the relations (37); for the equilibrium will not discontinue to exist if the profile assumed by this line of the material couch becomes the back face of a sustaining wall, and we shall have the solution of the problem of the equilibrium of a mass which, instead of being infinite, would be limited and sustained on one side by such a wall.

It is evident that the formulae (48), (49), (50), taken in all their generality, will hold true for whatever profile of a rough wall if the particles adjacent to this wall remain there immovable in such positions that their displacements u, v might have precisely the values (48); but I will confine myself in this paragraph to the study of the modes for which the simple conditions will be satisfied at all the points of a same line.

Whatever may be the direction of a wall at the back plane face, intersecting the top slope along a horizon which we may suppose to be chosen for the axis of z, it is easy to determine the arbitrary constants c, c', c'', c_1', c_1'' in such a manner that the first or the second line of the relations (37) may be satisfied on the whole extent of this face.

22. *Case of a Mass Limited by a Wall With a Plane and Rough Back Face.*

Suppose, in the first place, that the face in question is rough and directed along OM (fig. 1, p XXXIII), so as to maintain only the part AOM of the mass. The earth couch adjacent to OM will remain immovable while all the other parts of the mass displace. Now, after what has been said below the formulae (58), the only material couches in the infinite mass which suffer neither dilatation nor contraction are those whose inclination to the vertical is at each point equal to one of the values of ϵ given by equa-

:tion (56): one of these couches having to coincide with OM, we shall have a unique mode of equilibrium which may be admissible in taking, after the same equation (56),

(61) $$\varepsilon = GOM; \ c = 0, \ c' = tg\ (\omega - 2\varepsilon).$$

Thus the earth couch initially adjacent to OM does not suffer any deformation, and every other couch parallel to it and situated at a distance D from the back face OM of the wall, simple slides in its plane in moving in the sense from O to M by the quantity $\frac{D \sin\omega}{m \cos(\omega - 2\varepsilon)}$. The material point O not displacing, we must have $u=0$, $v=0$ for $x=0$, $y=0$ and consequently $c_1'=0$, $c_1''=0$. Lastly the constant c'' is determined in such a manner that the displacement u, for instance, vanishes all along the straight line OM which makes the angle $a+\varepsilon$ with the axis of y and whose equation is consequently $x-y \quad tg(a+\varepsilon)=0$: we thus find that $c'+c''=-tg(a+\varepsilon)$, or

$$-c'' = tg(a+\varepsilon) + c' = tg(a+\varepsilon) + tg(\omega - 2\varepsilon) = tg(\tfrac{\pi}{4} - \tfrac{\omega}{2} + \varepsilon) + tg(\omega - 2\varepsilon)$$
$$= \frac{1}{\cos(\omega - 2\varepsilon)}.$$

We observe that, *of all the modes of equilibrium of an infinite mass, there is one and only one for which a plane couch OM of the pulverulent matter remains immovable while the deformations go on: it is one for which the various constants* c, c', c'', c_1', c_1'' *have the values which we have just determined; in particular, the constant* c *is there zero, and the angular parameter* ε *thus invariable and characteristic of the mode of equilibrium is equal to the inclination of the immovable couch to the vertical. The settling produced by the weight of the mass takes place parallel to this couch (or to the back face of the wall), and for each particle of matter it is equal to the product of its distance* D *to the wall by the constant factor* $\frac{\sin\omega}{m \cos(\omega - 2\varepsilon)}$.

To find how much the settling caused by the weight of the mass diminishes the inclination of the top slope to the horizon initially equal to ω.

The perpendicular D dropped from the origin on every plane parallel to the immovable couch makes the angle ε with the horizon and consequently the angle $\omega - \varepsilon$ with the initial direction of the top slope and below the latter; whence it results that the initial distance, measured parallel to the wall, of the foot of this perpendicular to the top slope is equal to $D tg(\omega - \varepsilon)$. As it decreases, in virtue of the settling, by the value of the total displacement, the inclination of the top slope to the perpendicular D, initially equal to $\omega-\varepsilon$, decreases simultaneously by a small angle ξ, such that the difference

$$D[tg(\omega-\varepsilon)-tg(\omega-\varepsilon-\xi)], \text{ or sensibly } \frac{\xi D}{cos^2(\omega-\varepsilon)}$$

is exactly equal to the expression $D\frac{sin\omega}{m\,cos(\omega-2\varepsilon)}$ of the total displacement. The settling then results in diminishing the inclination ω of the top slope above the horizon by the quantity

$$(62) \qquad \xi=\frac{sin\omega\,cos^2(\omega-\varepsilon)}{m\,cos(\omega-2\varepsilon)}=\frac{sin\omega}{2m\,cos(\omega-2\varepsilon)}[1+cos2(\omega-\varepsilon)].$$

We shall have the occasion in the following Article to know by what angle ξ' the settling causes to turn about the origin O a material line taken in the mass starting from this origin and initially inclined to the plane OM of the immovable couch by half a right angle. The perpendicular D dropped from a point in this straight line to OM is evidently distant from the origin O by the quantity $Dtg\frac{\pi}{4}$ before, and by $Dtg(\frac{\pi}{4}+\xi')$ after, the displacements. The increase, nearly equal to $\frac{D\xi'}{cos^2\frac{\pi}{4}}$ or to $2D\xi'$, which this distance receives, represents exractly the displacement $\frac{sin\omega}{m\,cos(\omega-2\varepsilon)}D$ of the point under consideration. We have then

$$(62^{bis}) \qquad \xi'=\frac{sin\omega}{2m\,cos(\omega-2\varepsilon)}.$$

Observe that the angle made by the top slope OA with the material line now under question which was initially inclined to OM by one-fourth of a right angle decreases by the quantity

$$(62^{ter}) \qquad \xi-\xi'=\frac{sin\omega\,cos2(\omega-\varepsilon)}{2m\,cos(\omega-2\varepsilon)}.$$

23. *Case of a Mass Limited by a Wall with a Plane and Smooth Back Face.*

Suppose, in the second place, that the back plane face OM' of the wall (fig. 2) is smooth. Then the infinitely thin earth couch OM' can sustain the normal pressures only, which amounts to the fact that it contains at each of its points the direction of one of the two principal dilatations ∂_1, ∂_2. Now we have denoted by ε', at Art. 20, the inclination of any one of these dilatations to the horizon or that of the other to the vertical. Hence we may take

$$(63) \qquad \varepsilon'=GOM',$$

and $\varepsilon=\varepsilon'-\frac{\pi}{4}$ after the formula (59) in which the last term may be chosen negative or positive as we wish. Thus ε must be the same at all the points of OM', i. e., at all the distances l from OA as in the preceding Art., so that equation (56) must be put

$$c=0, \quad c'=tg(\omega-2\varepsilon)=-cotg(\omega-2\varepsilon').$$

Besides these, the couch OM' can slide in its plane but not to leave it. I shall consider only the motion which the particle O situated at the coordinate origin can suffer along OM', or, what comes to the same thing, I will suppose the axes Ox, Oy to be endowed with such a translation that the origin O always coincides with this particle. We shall have then $u=0$, $v=0$ for $x=0$, $y=0$, and whence to take $c_1'=0$, $c_1''=0$ in the formulae (48). It remains then, to satisfy completely the second line of the relations (37), to express that the points of the couch OM' displaces along its proper direction OM' inclined by $\alpha+\varepsilon$ to the positive y, i. e., to determine c'' such that we may have $\frac{u}{v}=\frac{x}{y}=tg(\alpha+\varepsilon')=tg\left(\frac{\pi}{4}-\frac{\omega-2\varepsilon'}{2}\right)$ at all these points Now the values (48) of u and v, in which we have already $c=0$, $c_1'=0$, $c_1''=0$, $c'=-cotg\ (\omega-2\varepsilon')$, change this last condition to

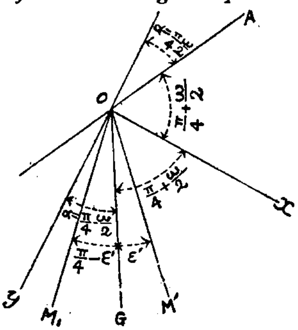
$$\frac{u}{v} \text{ or } tg\left(\frac{\pi}{4}-\frac{\omega-2\varepsilon'}{2}\right)=\frac{\frac{x}{y}+(c'+c'')}{-1+\frac{x}{y}(c'-c'')}=\frac{tg\left(\frac{\pi}{4}-\frac{\omega-2\varepsilon'}{2}\right)-cot(\omega-2\varepsilon')+c''}{-1-tg\left(\frac{\pi}{4}-\frac{\omega-2\varepsilon'}{2}\right)[cot(\omega-2\varepsilon)+c'']},$$

which, simplified by the substitution of $\frac{cos(\omega-2\varepsilon')}{1+sin(\omega-2\varepsilon')}$ for $tg\left(\frac{\pi}{4}-\frac{\omega-2\varepsilon'}{2}\right)$, and solved with respect to c'', finally gives

$$c''=0.$$

Among all the possible modes of equilibrium of an infinite mass, there is then one and only one for which a given plane couch of the pulverulent

(Fig. 2.)



matter can support the tangential stress and the displacements in its plane only: we obtain it by supposing the constant c to be zero and then by taking the angular parameter ε, *which remains to be solely characteristic af the equilibrium mode, to be equal to the inclination of this couch to the vertical decreased by* $45°$ *or* $\frac{\pi}{4}$. *This mode of equilibrium will exist if its every part situated at one side of the couch OM under consideration was replaced by an infinitely smooth sustaining wall, while the mass, becoming heavy after having been at first in the natural state, suffers the deformations as investigated.*

We have a clear idea of the settling produced in the actual case of a smooth wall having OM' for its back face, by conceiving, instead of a smooth wall, a rough one OM_1 inclined thereto by 45^0 or making the angle $yOM_1 = a + \varepsilon' - \frac{\pi}{4}$ with Oy, and by considering the settling then produced parallel to OM_1. This settling, at a distance D from OM_1, will be equal to

$$(64) \qquad D\frac{\sin\omega}{m\cos(\omega-2\varepsilon)} = -D\frac{\sin\omega}{m\sin(\omega-2\varepsilon')}.$$

To bring the mass AOM' to its definite state, it will be sufficient in consequence to turn it simply around the origin O in the sense from Oy towards Ox by the small quantity

$$(64^{bis}) \qquad \xi' = \frac{\sin\omega}{2m\cos(\omega-2\varepsilon)} = -\frac{\sin\omega}{2m\sin(\omega-2\varepsilon')},$$

in order to annul the equal and contrary rotation suffered in this fictitive settling, after (62^{bis}), by a material line initially couched against the real wall OM' and which really does not receive any rotation around O.

The inclination of the slope or of the surface OA to the wall OM' decreases in total by virtue of the settling and to conform to the formula (62^{ter}) by the small angle

$$(64^{ter}) \qquad \xi - \xi' = \frac{\sin\omega\cos2(\omega-\varepsilon)}{2m\cos(\omega-2\varepsilon)} = \frac{\sin\omega\sin2(\omega-\varepsilon')}{2m\sin(\omega-2\varepsilon')}.$$

24. *The Equilibrium Modes of the Infinite Mass Cannot be Satisfied by the Conditions* (37) *in Other Cases.*

The formulae (48), (49), (50) do not satisfy the first or the last two of the relations (37) along a line situated in the plane of xy, in any other case than those we have examined, i. e., in any case when the line under question will be curved.

This we easily recognize at first for the first two conditions (37). In fact, if we put $u=0$, $v=0$ in the formulae (48), the first becomes the equation of a parabola of the second degree whose axis is parallel to that of x, while the second becomes the equation of a parabola of the second degree having its axis parallel to that of y. These parabolas can evidently not coincide for a finite length unless they reduce to the straight lines or $c=0$, which brings us back to the case investigated at Art, 22.

Now to find how to satisfy the last two relations (37) at all the points of a same curved line.

If we put the values (49) of ∂_x, $-\partial_y$, ∂_{xy} in the formulae (33) of $sin2\beta_0$, $cos2\beta_0$, we find

$$sin2\beta_0=\frac{c'+cl}{\mp\sqrt{(c'+cl)^2+1}},\quad cos2\beta_0=\frac{1}{\mp\sqrt{(c'+cl)^2+1}},$$

where the radical must be taken with the same sign on either side. The first of the last relations (37) then gives

$$tg\gamma=-\frac{u}{v},$$

and consequently

$$cos2\gamma=\frac{v^2-u^2}{v^2+u^2},\quad sin2\gamma=\frac{-2uv}{v^2+u^2}.$$

The last equation of (37) which is nothing but

$$cos2\beta_0 sin2\gamma-sin2\beta_0 cos2\gamma=0,$$

then becomes

$$(65^{bis})\qquad (v^2-u^2)(c'+cl)+2uv=0.$$

It is sufficient to replace u, v, l herein by their values (48) and (39) to have the finite, integral and rational equation of the required line.

Again, γ denoting the angle made by the normal to this line with x, its angular coefficient $\frac{dy}{dx}$, is at each point equal to $-\frac{1}{tg\gamma}$ or to $\frac{v}{u}$, and we have

$$udy-vdx=0;$$

this differential equation, after the values (48) of u and v, can be immediately integrated. If we denote by c''' the arbitrary constant introduced by the integration, it becomes

$$(65^{ter})\ \tfrac{c}{3}(y^3sina-x^3cosa)+xy+\tfrac{c'}{2}(y^2-x^2)+\tfrac{c''}{2}(y^2+x^2)+c_1'y-c_1''x+c'''=0.$$

That the line (65^{ter}) which is of the third degree, may have a *curved* arc in common with the line of the fifth degree represented by (65^{bis}), it is necessary that the first members of these equations admit as a common factor a function of the two variables x, y and of at least of the second degree. In particular, the terms of the highest order which are, with the exception of the finite and constant coefficients,

$$c(x^3cosa-y^3sina)$$

for (65^{ter}), and, observing the values (48) and (39) of u, v, l,

$$c^3(xcosa+ysina)(x^4cos^2a-y^4sin^2a)$$

for (65^{bis}), must have a common factor of this degree, unless they are identically zero. As they do not admit a such, it is necessary that these terms vanish or that we have $c=0$. But then, after the formulae (56) and (59), the angles ε' which measure the inclination of the principal dilatations to the vertical are constant at all the points of the medium, and the surfaces, which suffer the normal stresses only or in which the last relation of (37) is satisfied on all its extent, are reduced to planes.