§ II.

GENERAL EXPRESSIONS OF THE ELASTIC FORCES AT THE INTERIOR OF SOLID OR PULVERULENT BODIES OF CONSTANT ELASTICITY.

9. *Expressions of Dilatations and Shears as the Functions of the Partial Derivatives of the Displacements.*

The formulae of the principal stresses F_1, F_2, F_3 as the functions of the principal dilatations ∂_1, ∂_2, ∂_3 being obtained, it remains to us to find those which express the six components, $N_1, N_2, N_3, T_1, T_2, T_3$, (Lamé's notations), of the stresses exerted upon the unit surface of the three plane elements perpendicular to three axes of rectangular coordinates x, y, z, as the functions of the three dilatations ∂_x, ∂_y, ∂_z of the material lines which were initially parallel to these axes and of the three cosines g_{yz}, g_{zx}, g_{xy} (*shears*) of the angles subtended, after the deformations, by the same lines taken two and two.

I will designate:

1° x, y, z the initial coordinates of the material particle from which these three small lines are taken to proceed and where the plane elements under consideration intersect one another;

2° u, v, w, continuous functions of x, y, z, the displacements along the axes, at the time t, of the same particle, that is the small increments received at this time by the initial coordinates x, y, z;

3° x', y', z' its initial coordinates with respect to another determinate system of rectangular axes having the same origin, and which I will suppose to be finally parallel to the three directions along which the principal dilatations ∂_1, ∂_2, ∂_3 at the particular point under consideration are produced;

4° u', v', w' the displacements of the same particle along these new axes;

5° lastly, $a, b, c; a', b', c'; a'', b'', c''$ the cosines of the angles which the axis of x, the axis of y, the axis of z make with these new axes of x', y', z'.

If dx, dy, dz denote the initial lengths of three infinitely small material lines initially parallel to the axes of x, y, z and taken to proceed from the particle under consideration, their projections on the axes, after the displacements, will respectively become, as we know:

$$(1+\frac{du}{dx})dx, \quad \frac{dv}{dx}dx, \quad \frac{dw}{dx}dx, \quad \text{for the first } dx,$$

$$\frac{du}{dy}dy, \quad (1+\frac{dv}{dy})dy, \quad \frac{dw}{dy}dy, \quad \text{for the second } dy,$$

$$\frac{du}{dz}dz, \quad \frac{dv}{dz}dz, \quad (1+\frac{dw}{dz})dz, \quad \text{for the third } dz.$$

Consequently, the dilatations of these lines, the excess of the ratios of their actual lengths to their initial lengths over the unity, will be sensibly equal to

$$(15). \qquad \partial_x=\frac{du}{dx}, \quad \partial_y=\frac{dv}{dy}, \quad \partial_z=\frac{dw}{dz},$$

and the cosines of the angles made by them two and two will be, to the same degree of approximation,

$$(16). \qquad g_{yz}=\frac{dv}{dz}+\frac{dw}{dy}, \quad y_{zx}=\frac{dw}{dx}+\frac{du}{dz}, \quad g_{xy}=\frac{du}{dy}+\frac{dv}{dx}.$$

10. *Formulae for the Transformations of Coordinates:* 1° *Transformation of the Dilatations and Shears.*

The known formulae of the transformation of the coordinates moreover give

$$x=ax'+by'+cz', \qquad y=a'x'+b'y'+c'z', \qquad z=a''x'+b''y'+c''z';$$

$$\frac{d}{dx}=a\frac{d}{dx'}+b\frac{d}{dy'}+c\frac{d}{dz'}, \quad \frac{d}{dy}=a'\frac{d}{dx'}+b'\frac{d}{dy'}+c'\frac{d}{dz'}, \quad \frac{d}{dz}=a''\frac{d}{dx'}+b''\frac{d}{dy'}+c''\frac{d}{dz'};$$

$$u=au'+bv'+cw', \qquad v=a'u'+b'v'+b'w', \qquad w=a''u'+b''v'+c''w'.$$

We therefrom immediately deduce the following relations

$$\frac{du}{dx}=a^2\frac{du'}{dx'}+b^2\frac{dv'}{dy'}+c^2\frac{dw'}{dz'}+bc\left(\frac{dv'}{dz'}+\frac{dw'}{dy'}\right)+ca\left(\frac{dw'}{dx'}+\frac{du'}{dz'}\right)+ab\left(\frac{du'}{dy'}+\frac{dv'}{dx'}\right).$$

$$\frac{dv}{dz}+\frac{dw}{dy}=2\left(a'a''\frac{du'}{dx'}+b'b''\frac{dv'}{dy'}+c'c''\frac{dw'}{dz'}\right)+(b'c''+c'b'')\left(\frac{dv'}{dz'}+\frac{dw'}{dy'}\right)$$

$$+(c'a''+a'c'')\left(\frac{dw'}{dx'}+\frac{du'}{dz'}\right)+(a'b''+b'a'')\left(\frac{du'}{dy'}+\frac{dv'}{dx'}\right);$$

$$\frac{dv}{dy}=\cdots\cdots\cdots, \quad \frac{dw}{dx}+\frac{du}{dz}=\cdots\cdots\cdots; \quad \frac{dw}{dz}=\cdots\cdots\cdots, \quad \frac{du}{dy}+\frac{dv}{dx}=\cdots\cdots\cdots.$$

Now, by the hypothesis, the axes of x', y', z' are parallel to the three *principal* directions, for which, at the particular point (x, y, z), the shears or cosines $dv'/dz'+dw'/dy'$, $dw'/dx'+du'/dy'$, $du'/dy'+dv'/dx'$ are zero, while the dilatations du'/dx', dv'/dy', dw'/dz' have been called ∂_1, ∂_2, ∂_3. The preceding formulae then simply become:

$$(17).\quad \begin{cases} \partial_x = a^2\partial_1 + b^2\partial_2 + c^2\partial_3, \quad \partial_y = a'^2\partial_1 + b'^2\partial_2 + c'^2\partial_3, \quad \partial_z = a''^2\partial_1 + b''^2\partial_2 + c''^2\partial_3; \\ g_{yz} = 2(a'a''\partial_1 + b'b''\partial_2 + c'c''\partial_3), \qquad g_{zx} = 2(a''a\partial_1 + b''b\partial_2 + c''c\partial_3), \\ g_{xx} = 2(aa'\partial_1 + bb'\partial_2 + cc'\partial_3). \end{cases}$$

Many useful consequences result from these formulae. Firstly, if we add the three expressions of ∂_x, ∂_y, ∂_z, it becomes, by virtue of the known relations which exist between the cosines a, a', a'',......,

$$(18).\qquad \partial_x + \partial_y + \partial_z = \partial_1 + \partial_2 + \partial_3 = \theta \text{ in consequence.}$$

Again, if we subtract ∂_z from ∂_y, for instance, and if we replace, in the result, a'^2, a''^2 by $1 - b'^2 - c'^2$, $1 - b''^2 - c''^2$, we find

$$(19).\quad \begin{cases} \partial_y - \partial_z = (c'^2 - c''^2)(\partial_3 - \partial_1) - (b'^2 - b''^2)(\partial_1 - \partial_2); \\ \text{we shall have similarly} \\ \partial_z - \partial_x = (a''^2 - a^2)(\partial_1 - \partial_2) - (c''^2 - c^2)(\partial_2 - \partial_3), \\ \partial_x - \partial_y = (b^2 - b'^2)(\partial_2 - \partial_3) - (a^2 - a'^2)(\partial_3 - \partial_1). \end{cases}$$

Finally, by eliminating $a'a''$ from the expression (17) of g_{yz} by means of the relation

$$a'a'' + b'b'' + c'c'' = 0,$$

it becomes

$$(20).\quad \begin{cases} g_{yz} = 2c'c''(\partial_3 - \partial_1) - 2b'b''(\partial_1 - \partial_2); \\ \text{we shall have similarly} \\ g_{zx} = 2a''a(\partial_1 - \partial_2) - 2c''c(\partial_2 - \partial_3); \\ g_{xy} = 2bb'(\partial_2 - \partial_3) - 2aa'(\partial_3 - \partial_1). \end{cases}$$

11. 2⁰ *Transformation of the Elastic Forces.*

On the other hand, the well-known relations, deduced from the consideration of the dynamical equilibrium of the elementary tetrahedron of Cauchy furnish the required formulae for the transformation of the stresses N, T. If p_x, p_y, p_z denote the components, along the three rectangular axes of x, y, z, of the stress exerted on the unit of area of any plane element whose normal makes the angles α, β, γ with these axes, these relations are:

$$(21)\quad \begin{cases} p_x = N_1\cos\alpha + T_3\cos\beta + T_2\cos\gamma, \\ p_y = T_3\cos\alpha + N_2\cos\beta + T_1\cos\gamma, \\ p_z = T_2\cos\alpha + T_1\cos\beta + N_3\cos\gamma. \end{cases}$$

When applied to the components, p'_x, p'_y, p'_z, along the particular system of axes of x', y', z' for which the normal forces N become F_1, F_2, F_3 while the tangential forces T are zero, these formulae are reduced to

$$(21^{bis}).\qquad p_{x'} = F_1\cos\alpha', \quad p_{y'} = F_2\cos\beta', \quad p_{z'} = F_3\cos\gamma',$$

where I denote by α', β', γ' the angles made with these axes by the normal to the plane element under consideration.

We shall have the components, always along the axes of x', y', z', of the stresses exerted upon the plane elements normal to the axes of x, y, z, by making successively in (21^{bis}):

$cos\alpha' = a, = a', = a''$; $cos\beta' = b, = b', = b''$; $cos\gamma = c, = c', = c''$.

The projections N_1, T_3, T_2; T_3, N_2, T_1; T_2, T_1, N_3, along x, y, z of the same stresses, will consequently be obtained by taking the sum of these components, multiplied respectivelly by a, b, c or a', b', c' or a'', b'', c''. It thus becomes

$$(22).\quad \begin{cases} N_1 = a^2F_1 + b^2F_2 + c^2F_3,\ N_2 = a'^2F_1 + b'^2F_2 + c'^2F_3,\ N_3 = a''^2F_1 + b''^2F_2 + c''^2F_3; \\ T_1 = a'a''F_1 + b'b''F_2 + c'c''F_3, \quad T_2 = a''aF_1 + b''bF_2 + c''cF_3, \\ T_3 = aa'F_1 + bb'F_2 + cc'F_3. \end{cases}$$

These formulae are similar to (17), and we shall similarly deduce therefrom the relations analogous to (18), (19), (20):

$$(23).\quad \begin{cases} N_1 + N_2 + N_3 = F_1 + F_2 + F_3 = -3p \text{ in consequence;} \\ N_2 - N_3 = (c'^2 - c''^2)(F_3 - F_1) - (b'^2 - b''^2)(F_1 - F_2), N_3 - N_1 = \cdots, N_1 - N_2 = \cdots; \\ T_1 = c'c''(F_3 - F_1) - b'b''(F_1 - F_2),\ T_2 = \cdots,\ T_3 = \cdots. \end{cases}$$

12. *Formulae of the Elastic Forces for the Isotropic Solids and the Pulverulent Media.*

It is sufficient to compare respectively the expressions (23) of $N_2 - N_3$ and of T_1 to those (19) of $\partial_y - \partial_z$ and (20) of g_{yz}, taking account of the continued equality (14), to find that we have

$$\frac{\frac{1}{2}(N_2 - N_3)}{\partial_y - \partial_z} = \frac{T_1}{g_{yz}} = \begin{cases} \mu \text{ (if the body is solid),} \\ mp \text{ (if it is pulverulent);} \end{cases}$$

the ratios

$$\frac{\frac{1}{2}(N_3 - N_1)}{\partial_z - \partial_x}, \quad \frac{T_2}{g_{zx}}, \quad \frac{\frac{1}{2}(N_1 - N_2)}{\partial_x - \partial_y}, \quad \frac{T_3}{g_{xy}},$$

are similarly equal to μ or mp.

From this, result immediately the required values of T_1, T_2, T_3 as well as those of the differences $N_2 - N_3$, $N_3 - N_1$, $N_1 - N_2$. As to the expressions themselves of N_1, N_2, N_3, the identities whose first is

$$N_1 = \tfrac{1}{3}(N_1 + N_2 + N_3) - \tfrac{1}{3}(N_3 - N_1) + \tfrac{1}{3}(N_1 - N_2)$$

give them, when we observe that, after a relation (23), the mean normal force $\frac{1}{3}(N_1 + N_2 + N_3)$ is equal to the arithmetical mean $-p$ of the three principal stresses and have for its value, in the case of an isotropic solid (see form. 5), $A + (\lambda + \frac{2}{3}\mu)\theta$. Taking account of (18) we thus find, in this case, the well-known formulae:

$$(24).\quad \begin{cases} N_1 = A + \lambda\theta + 2\mu\partial_x,\ N_2 = A + \lambda\theta + 2\mu\partial_y,\ N_3 = A + \lambda\theta + 2\mu\partial_z, \\ T_1 = \mu g_{yz}, \quad T_2 = \mu g_{zx}, \quad T_3 = \mu g_{xy}, \\ \text{where, after (18), } \theta = \partial_x + \partial_y + \partial_z. \end{cases}$$

If, on the contrary, the body is pulverulent, the case in which we have seen that the cubical dilatation θ can be neglected, it becomes

$$(25).\quad \begin{cases} N_1=-p(1-2m\partial_x),\; N_2=-p(1-2m\partial_y),\; N_3=-p(1-2m\partial_z),\\ T_1=pmg_{yz},\qquad T_2=pmg_{zx},\qquad T_3=pmg_{xy},\\ \text{with the condition } \theta \text{ or } \partial_x+\partial_y+\partial_z=0. \end{cases}$$

We shall be able, in either case. to substitute to the six deformations ∂_x, ∂_y, ∂_z, g_{yz}, g_{zx}, g_{xy} their approximate expressions (15) and (16) as the functions of the partial derivatives of the displacements u, v, w with respect to the initial coordinates x, y, z.

§ III.

DIFFERENTIAL EQUATIONS OF THE ELASTIC EQUILIBRIUM OF THE PULVERULENT MASSES.

13. *Preliminary Considerations.*

I shall principally be occupied, in the rest of this study, with the equilibrium of heavy masses, such as a heap of sand, formed of very small juxtaposited solid grains without cohesion but mutually compressed. We shall be able to be abstracted of the atmospheric pressure, applied normally to each plane element taken amid the mass; because this pressure, which exists in all the senses at the interior and all around the various grains, even before we bring them together and also in the air interposed, has no effect on keeping them pressed one another and consequently by no means modifies the two supplementary stresses, normal and tangential, which is generally produced by the contact of the grains on the unit of area of the plane element. These supplementary stresses will be the only one which we shall have to consider, and to which we shall apply the above established formulae and notably the general relations (21).

We shall suppose at first the mass to be weightless, free of any stress at each point, and we shall take for initial coordinates x, y, z of its various particles with respect to a system of fixed rectangular axes, the coordinates which they shall have in this state of repose, called *natural state* (*état naturel*); then we shall conceive that they become heavy and we shall propose to determine the small displacements u, v, w which shall be undergone by its various parts when a new equilibrium, which we shall suppose at first to be possible in these conditions, will be established.

If the elastic limits of the mass are not surpassed, as we shall admit it, the formulae (25), special to the pulverulent bodies must here be applied; because the hypotheses, adopted to establish them, of an equal constitution in all directions and of a rigidity zero or finite according as the mean stress is the such in itself, precisely apply to the masses under question. The expreience shows that the tangential actions T are generally of the order of magnitude of normal actions N or of the mean stress p; whence it follows that the numerical positive coefficient m is sufficiently large that its products by the small deformations ∂, g have the values comparable to unity. We shall suppose moreover m constant, which amounts to admit the homogeneity of the mass, that is the parity of composition of all its parts of considerable extent.

14. *Indefinite Equations of the Equilibrium—Case of the Uniplanar Deformations.*

To find at first the indefinite equations of the equilibrium. All the terms contained in the expressions (25) of the forces N, T have linearly as a factor the small deformations ∂, g with the exception of the term $-p$, which appears in N_1, N_2, N_3 but which is solely comparable to others as it is to be. We can then, in comparison with the derivatives in x, y, z of the forces N, T, neglect the products of these derivatives by those of the small displacements u, v, w and deduce in consequence, as we ordinarily do in the elastic theory of the solid bodies, the conditions which express the equilibrium of translation of an element of rectangular volume to the known formulae

$$(26).\quad \begin{cases} \frac{dN_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}+\rho X=0, \qquad \frac{dT_3}{dx}+\frac{dN_2}{dy}+\frac{dT_1}{dz}+\rho Y=0, \\ \frac{dT_2}{dx}+\frac{dT_1}{dy}+\frac{dN_3}{dz}+\rho Z=0, \end{cases}$$

where ρ denote the density of the mass, that is its mass of the unit of *apparent* volume, and X, Y, Z the components of the gravity g along the three axes of x, y, z.*

I shall be confined to the theory of deformations parallel to a vertical plane, taken for one of xy and in which the displacement w will be zero while the displacements u, v will depend only on x and y. The relations (15), (16), (25) will be reduced by these hypotheses to these:

$$(27).\quad \begin{cases} \partial_x=\frac{du}{dx}, \quad \partial_y=\frac{dv}{dy}, \quad g_{xy}=\frac{du}{dy}+\frac{dv}{dx}, \qquad \partial_z=0, \; g_{yz}=0, \; g_{zx}=0; \\ N_1=-p(1-2m\partial_x), \; N_2=-p(1-2m\partial_y), \; N_3=-p; \\ T_1=0, \qquad T_2=0, \qquad T_3=pmg_{xy}; \\ \partial_x+\partial_y=0 \end{cases}$$

We find that the traction applied to the surface elements parallel to the plane of xy, the traction which has the components T_2, T_1, N_3 along the axes of x, y, z respectively, is normal to these plane elements and is equal to $-p$: in other words, *it exerts at each point, on the surface element parallel to the planes of the deformations, a simple normal stress equal to the mean stress produced at this point.*

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* We can find, in § I of the *Theory of the periodic liquid waves* (*Théorie des ondes liquides périodiques*, RECUEIL DES SAVANTS ÉTRANGERS DE L'ACADÉMIE DES SCIENCES DE PARIS, t. XX, 1872), the more general formulae which must be substituted to these equations (26), if the derivatives of the forces N, T in x, y, z contain the terms of a considerable magnitude indepedendent of the small deformations ∂, g as it precisely happens in the case under question of liquid waves.

If α and $\frac{\pi}{2}-\alpha$ denote the angles which the gravity makes with the two axes of y and x, we have $X=g\sin\alpha$, $Y=g\cos\alpha$, $Z=0$ and the first two equations of (26) become

$$(28).\qquad \frac{dN_1}{dx}+\frac{dT}{dy}+\rho g\sin\alpha=0,\ \frac{dT}{dx}+\frac{dN_2}{dy}+\rho g\cos\alpha=0,$$

in which I can simply write T for T_3, as I shall do hereafter, since it has no more occasion to be occupied either with T_1 or T_2.

As to the third of (26), it is reduced to

$$\frac{dN_3}{dz}=0,\qquad \text{or}\qquad -\frac{dp}{dz}=0,$$

and signifies, what is evident, that the mean stress p, as u and v, depends only on x and y

In general, we shall substitute, in (28), for N_1, N_2, T their expressions (27), so as to leave u, v, p only as unknown. But we may have sometimes the need to determine directly the forces N_1, N_2, T without being occupied with the displacements u, v. Then a third equation in N_1, N_2, T becomes necessary. It is deduced from the identity

$$\frac{d^2}{dxdy}\left(\frac{du}{dy}+\frac{dv}{dx}\right)=\frac{d^2}{dx^2}\frac{dv}{dy}+\frac{d^2}{dy^2}\frac{du}{dx},\quad \text{or}\quad \frac{d^2g_{xy}}{dxdy}=\frac{d^2\partial_y}{dx^2}+\frac{d^2\partial_x}{dy^2},$$

by putting therein the values

$$g_{xy}=\frac{1}{2m}\frac{2T}{p},\ \partial_y=\frac{1}{2m}\left(\frac{N_2}{p}+1\right)=\frac{1}{2m}\left(\frac{N_2-N_1}{2p}\right),\ \partial_x=\frac{1}{2m}\left(\frac{N_1-N_2}{2p}\right),$$

which are furnished by three of the relations (27) for the deformations g_{xy}, ∂_y, ∂_x: it thus becomes

$$(28^{bis}).\ .\ 2\frac{d^2}{dxdy}\left(\frac{T}{p}\right)=\left(\frac{d^2}{dx^2}-\frac{d^2}{dy^2}\right)\left(\frac{N_2-N_1}{2p}\right),\quad \text{where } p=-\tfrac{1}{2}(N_2+N_1).^{*}$$

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* When the body, supposed always to be homogeneous and isotropic, is solid and non-pulverulent, three of the formulae (24) give, in making $\partial_z=0$, $\theta=\partial_x+\partial_y$,

$$g_{xy}=\frac{1}{2\mu}.2T,\ \partial_y=\frac{1}{2\mu}\left[N_2-A-\frac{\lambda}{\lambda+\mu}\left(\frac{N_2+N_1}{2}-A\right)\right],\ \partial_x=\frac{1}{2\mu}\left[N_1-A-\frac{\lambda}{\lambda+\mu}\left(\frac{N_2+N_1}{2}-A\right)\right].$$

The equation (28^{bis}) must then be replaced by this

$$(28^{ter}).\qquad 2\frac{d^2T}{dxdy}=\frac{d^2N_2}{dx^2}+\frac{d^2N_1}{dy^2}-\frac{\lambda}{2(\lambda+\mu)}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)(N_2+N_1).$$

If ϖ denote an unknown function, which will only remain to be determined, the two equations (28) become to be put, as we know easily,

$$T=-\rho g\frac{d^2\varpi}{dxdy},\ N_2=A+\rho g\left(\frac{d^2\varpi}{dx^2}-y\cos\alpha\right),\ N_1=A+\rho g\left(\frac{d^2\varpi}{dy^2}+x\sin\alpha\right),$$

and the formulae (28^{ter}) takes the form

$$\Delta_2\Delta_2\varpi=0,$$

in representing by Δ_2 the symbolical expression $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}$.

15. *Stresses, Dilatations and Shears Parallel to the Plane of Deformations.*

We shall have subsequently the need to know the elastic force exerted on the plane element, parallel to the axis of z, whose normal will make, with those of x and y, two given angles β and $\frac{\pi}{2}-\beta$. The general formulae (21), in which it will thus be necessary to replace $cos a$, $cos\beta$, $cos\gamma$ by $cos\beta$, $sin\beta$, o and besides to put $T_2=0$, $T_1=0$, will give, for the components of this stress respectively along x, y and z,

(29). $\quad p_x=N_1 cos\beta+T sin\beta,\ p_y=T cos\beta+N_2 sin\beta,\ p_z=0.$

But it is preferable to have its two components, *normal*, $\mathfrak{N}$, and *tangential*, $\mathfrak{T}$ (along the direction which makes the angle $\beta+\frac{\pi}{2}$ with the positive x): we shall obtain them by adding the components (29) after having multiplied them respectively by the cosines of the angles made by the direction on which we project them with x, y, z, namely, for $\mathfrak{N}$ by $cos\beta$, $sin\beta$, o, and for $\mathfrak{T}$ by $cos(\beta+\frac{\pi}{2})$, $sin(\beta+\frac{\pi}{2})$, o or by $-sin\beta$, $cos\beta$, o. Thus, after having replaced $cos^2\beta$, $sin^2\beta$, $cos\beta sin\beta$ by $\frac{1}{2}(1-cos2\beta)$, $\frac{1}{2}(1-cos2\beta)$, $\frac{1}{2}\ sin2\beta$ and observing that $N_1+N_2=-2p$, it becomes:

$$(30).\qquad \mathfrak{N}=-p-\frac{N_2-N_1}{2}cos2\beta+T sin2\beta,\ \mathfrak{T}=\frac{N_2-N_1}{2}siu2\beta+T cos2\beta.$$

I will designate by β_0 the auxiliary angle comprised between zero and π and defined by the relations

$$(31).\qquad \begin{cases} sin^2\beta_0=-\dfrac{T}{R},\quad cos2\beta_0=\dfrac{\frac{1}{2}(N_2-N_1)}{R}, \\ \text{where } R=\sqrt{T^2+\left(\dfrac{N_2-N_1}{2}\right)^2}; \end{cases}$$

by replacing then T and $\frac{1}{2}(N_2-N_1)$ by $-R sin2\beta_0$ and $R cos2\beta_0$, the formulae (30) will simply become

(32). $\quad \mathfrak{N}=-p-R cos2(\beta-\beta_0),\qquad \mathfrak{T}=R sin2(\beta-\beta_0).$

We can besides, in the formulae (31), substitute for N_1, N_2, T their values (27); it gives

$$(33).\quad \begin{cases} \sin 2\beta_0 = \dfrac{-g_{xy}}{\pm\sqrt{g_{xy}^2+(\partial_y-\partial_x)^2}}, & \cos 2\beta_0 = \dfrac{\partial_y-\partial_x}{\pm\sqrt{g_{xy}^2+(\partial_y-\partial_x)^2}}, \\ R = \pm mp\sqrt{g_{xy}^2+(\partial_y-\partial_x)^2}, \end{cases}$$

where upper or lower signs are to be taken according as the mean stress p will be positive or negative.

It will be further useful to us to know, as a function of ∂_x, ∂_y, g_{xy}, 1^0 the dilatation $\partial_{x'}$ of the material line, parallel to the plane of xy and initially making the angles β, $\frac{\pi}{2}-\beta$, o with the axes of x, y, z respectively, 2^0 the dilatation $\partial_{y'}$ of the line initially normal thereto and equally parallel to xy and which makes the angle $\beta+\frac{\pi}{2}$ with the axis of x, and 3^0 finally the small cosine $g_{x'y'}$ of the angle made, after the deformations, by these two material lines. It is sufficient for this to observe that the expressions (27) of N_1, T_3, applicable to all the systems of rectangular axes taken in the plane of xy and especially to those which would have been initially parallel to these lines, give

$$\mathfrak{N} = -p(1-2m\partial_{x'}), \qquad \mathfrak{T} = pmg_{x'y'}.$$

From the comparison of these values of $\mathfrak{N}$, $\mathfrak{T}$ to (32), and observing the value (33) of R, it results:

$$(34).\quad \begin{cases} \partial_{x'} = -\dfrac{R}{2mp}\cos 2(\beta-\beta_0) = \mp\tfrac{1}{2}\sqrt{g_{xy}^2+(\partial_y-\partial_x)^2}\cos 2(\beta-\beta_0), \\ g_{x'y'} = \dfrac{R}{mp}\sin 2(\beta-\beta_0) = \pm\sqrt{g_{xy}^2+(\partial_y-\partial_x)^2}\sin 2(\beta-\beta_0). \end{cases}$$

On the other hand, the last formula of (27), equally applicable to all the systems of rectangular axes taken in the plane of xy, gives

$$(34^{bis}).\qquad \partial_{y'} = -\partial_{x'}.$$

The three principal stresses F_1, F_2, F_3 are obtained easily by means of the relations (27) and (32). Firstly, one of them, parallel to the axis of z, is N_3 or $-p$ by reason of symmetry. The two others, evidently exerted on two plane elements parallel to the axis of z, are the two values of $\mathfrak{N}$ which correspond to the zero values of $\mathfrak{T}$, that is to $\sin 2(\beta-\beta_0)=0$, $\cos 2(\beta-\beta_0)=\pm 1$; they are $-p+R$ and $-p-R$. Arranging these three stresses in the descending order of magnitude, it thus becomes

$$(34^{ter}).\qquad F_1=-p+R,\ F_2=-p,\ F_3=-p-R.$$

The three corresponding principal dilatations are consequently

$$(34^{quater})\qquad \partial_1=\frac{R}{2mp},\qquad \partial_2=0,\qquad \partial_3=-\frac{R}{2mp}.$$

Finally observe that the two principal forces F_1, F_2 make, with the straight line whose inclination to the axis of x is denoted by β_0, two angles

$\beta-\beta_0$, such that $cos2(\beta-\beta_0)=-1$ for F_1, $cos2(\beta-\beta_0)=+1$ for F_3; these angles are then equal to $\frac{\pi}{2}$ for F_1 and zero for F_3, putting aside an integral number of semi-circumference. Thus, *the least F_3 of the principal elastic forces coincide with the direction which makes the angle β_0 with the positive x.*

16. *Special Conditions at the Boundary Surface.*

If the small displacements u, v, w and the mean stress p vary continuously from point to point in *all* the extent of the mass (as I will admit), there will be no other equation of equilibrium to be satisfied than the condition of incompressibility

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0,$$

and the three indefinite equations (26), in which we suppose the forces N, T to be replaced by their expressions as functions of u, v, w, p. But there will exist some special conditions relating, on one hand to the free surfaces where the pulverulent mass is related only with the atmosphere, and, on the other, to the wall surfaces which will be constituted either of a solid ground or of the back faces of the retaining walls.

At the free surfaces, it must be expressed that the stress exerted by the mass upon its surface couch has three components along the axes equal and opposite to those of the stress exerted from without on the same couch, and to the zero stress when we have been abstracted from the atmosphere. The third of these conditions is verified identically in the case of uniplanar deformations to which we are confined: in fact, at the point of the free surface where the normal to this surface taken towards the interior will make the angles γ, $\frac{\pi}{2}-\gamma$, 0 with the axes of x, y, z respectively, the formulae (29), in which it is sufficient to replace β by γ and to put $p_x=0$, $p_y=0$, $p_z=0$, simply give

(35). $N_1 cos\gamma+T sin\gamma=0$, $T cos\gamma+N_2 sin\gamma=0$ (at the free surface).

At the walls, there will be equally three special conditions. The first will be obtained by expressing that the points of the mass adjacent to a wall do not part therewith, or that their coordinates $x+u$, $y+v$, $z+w$ satisfy constantly the equation of the surface. The two others will vary with the degree of roughness of the wall. I will consider only two extreme cases, in which this degree will be either sufficient to prevent any finite sliding of the adjacent couch of the mass or on the contrary zero, that is the such that the wall does not exert any friction or any sensible tangential action. In order to discuss in the first place the simplest problems, I will also suppose at first that the mass, after having been put on the ground which supports it and against the walls which sustain it, may be found for an instant in the *natural* state before becoming heavy as it really is, and besides I will admit the absolute immobility of the retaining walls.

In the case of a rough wall, the two components of the displacement along two rectangular directions taken tangentially to the wall as well as its component normal to it will be equal to zero, and we shall have there definitively $u=0$, $v=0$, $w=0$. In the contrary case of a smooth wall the two components, along these rectangular directions tangential to the surface, of the action exerted by the mass upon its surface couch must vanish. We restrict ourselves always to the study of uniplanar deformations and denote moreover by γ, $\frac{\pi}{2}-\gamma$, 0 the angles which the normal to an element of the surface of the mass taken towards its interior makes with the axes of x, y, z respectively. The formulae (32), if we replace therein β by γ, will give for the two components, normal $-\mathfrak{N}$ and tangential $\mathfrak{T}$, of the pressure exerted by the mass on the unit of area of its surface couch (or consequently of the adjacent wall):

(36). $$-\mathfrak{N}=p+R\cos 2(\gamma-\beta_0),\quad \mathfrak{T}=R\sin 2(\gamma-\beta_0),$$

where R and β_0 will have the values defined by the relations (33). The radical R being essentially positive, we shall in general annul $\mathfrak{T}$ by putting $\sin 2(\gamma-\beta_0)=0$ when the wall is perfectly smooth: in the same case, the component $u\cos\gamma+v\sin\gamma$, normal to the wall, of the displacement of the surface couch ought to be zero. The required conditions will be then:

(37). $$\begin{cases} u=0,\ v=0, \text{ (for a rough wall)}, \\ u\cos\gamma+v\sin\gamma=0,\ \sin 2(\gamma-\beta_0)=0, \text{ (for a smooth wall)}. \end{cases}$$

It must not be forgotten that these conditions at the walls, the simplest which one can imagine, refer only to the hypothetical case in which the *natural* state, wherefor we have everywhere $u=0$, $v=0$, $w=0$, will exist previous to the mode of equilibrium as studied: in other words, they only apply so far as the pulverulent mass is supposed to be at first free of every stress and weightless, and then to be deformed under the action of its effective weight, without that the couch adjacent to a wall suffers any displacent, in the normal direction when it is smooth, and in any direction when it is rough. But these circumstances are not realised in practice. The walls which we really construct have no doubt their back faces more rough than it requires to prevent the sliding of the adjacent particles of earth, but these may be found at rest in other positions than those of natural state, and the required conditions will be obtained by equating their displacements u, v, w to some functions of x, y, z which must be supposed to be given in each case that the problem of the equilibrium may be determinate, but which will be really unknown. Similarly, in a smooth wall, the total component of the displacements along the normal to the wall will generally be a determinate function in each case, although unknown, of x, y, z and will only exceptionally vanish.

With the data prepared by the engineer, the equilibrium which is produced in the mass, at the moment itself in which we form it by discharging successively the earth on the ground or against a retaining wall, does not seem to be susceptible of a precise determination, and it must be very complex or affected by a great number of local anomalies.* But what is of moment to know, is the mode of *definitive* equilibrium which will be in existance, when the small shocks which every mass undergoes nearly at each instant will make the irregularities disappear and bring a complete settling or group all the sandy grains somewhat in the least constrained manner. Such a mode of equilibrium, by the fact itself that it takes it up in preferance to any other, must be, *of all the modes compatible to the circumstances, that which assure the best internal stability of the mass in deviating the least possible from the natural state.* We shall see, in §VIII, how this condition of stability can take place from the acquaintance of the special relations at the wall.

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* The same difficulty will present itself if the mass be solid and elastic instead of being destitute of cohesion: it results only from our ignorance of the real conditions in which the couch adjacent to the wall is found, and by no means of the theory itself of the elastic equilibrium of the pulverulent masses.