Notes on Tchebycheff's and Simpson's Rules.

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I. *Extension of Tchebycheff's rule to the calculation of the moment and the moment of inertia of a plane figure about an axis.*

1. The advantages of using Tchebycheff's rule for quadrature and cubature are very well known among us. So I have tried to extend this rule to the calculation of the moment and moment of inertia, and have obtained the following rules.

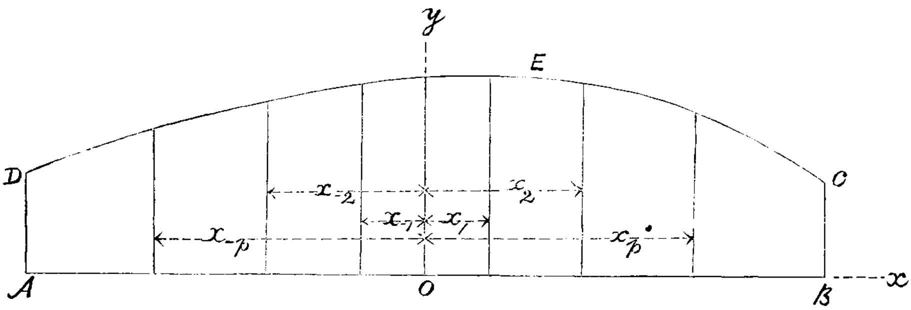


Fig. 1.

Let ABCED (Fig. 1) be an area. Take Oy drawn at the middle of the length perpendicular to the base AB for y axis, and OB for x axis. Assume the curve DEC to be a part of

$$y=a_0+a_1x+a_2x^2+a_3x^3+\cdots\cdots\cdots+a_{n-1}x^{n-1}+a_nx^n.$$

Then the moment (M) of the area about y axis

$$=\int_{-l}^{l}yxdx,$$

where l is the half length of the base AB.

$\therefore$

(1) $$M \int_{-l}^{l}(a_0x+a_1x^2+a_2x^3+\cdots\cdots\cdots+a_{n-1}x^n+a_nx^{n+1})dx$$

$$=2l^3\left(\frac{a_1}{3}+\frac{a_3}{5}l^2+\cdots\cdots\cdots\cdots\cdots\cdots+\frac{a_{n-2}}{n}l^{n-3}+\frac{a_n}{n+2}l^{n-1}\right), n \text{ odd};$$

$$=2l^3\left(\frac{a_1}{3}+\frac{a_3}{5}l^2+\cdots\cdots\cdots\cdots\cdots\cdots+\frac{a_{n-3}}{n-1}l^{n-4}+\frac{a_{n-1}}{n+1}l^{n-2}\right), n \text{ even}.$$

i) Consider the case when n is an odd integer

Let

$$M=c\left\{y_1+y_2+\cdots\cdots+y_{\frac{n-1}{2}}-(y_{-1}+y_{-2}+\cdots\cdots+y_{-\frac{n-1}{2}})\right\},$$

where $y_1, y_2,\cdots\cdots, y_{\frac{n-1}{2}}, y_{-1}, y_{-2},\cdots\cdots, y_{-\frac{n-1}{2}}$ are the lengths of ordinates corresponding to $x_1, x_2,\cdots\cdots, x_{\frac{n-1}{2}}, x_{-1}, x_{-2},\cdots\cdots x_{-\frac{n-1}{2}}$ respectively, and c is a constant.

Or,

$$(2)\qquad M=2c\left\{a_1(x_1+x_2+\cdots\cdots+x_{\frac{n-1}{2}})+a_3(x_1^3+x_2^3+\cdots\cdots+x^3_{\frac{n-1}{2}})+\cdots\cdots\cdots\cdots+a_n(x_1^n+x_2^n+\cdots\cdots+x^n_{\frac{n-1}{2}})\right\}.$$

Equating the coefficients of $a_1, a_3,\cdots\cdots,a_n$ in (1) and (2), we have the following $\frac{n+1}{2}$ equations:

$$(3)\cdots\cdots\cdots\cdots\left\{\begin{aligned}&c\,(x_1+x_2+\cdots\cdots+x_{\frac{n-1}{2}})=\frac{l^3}{3},\\ &c\,(x_1^3+x_2^3+\cdots\cdots+x^3_{\frac{n-1}{2}})=\frac{l^5}{5},\\ &\cdots\cdots\cdots\cdots\cdots\cdots=\cdots,\\ &\cdots\cdots\cdots\cdots\cdots\cdots=\cdots,\\ &c\,(x_1^n+x_2^n+\cdots+x^n_{\frac{n-1}{2}})=\frac{l^{n+2}}{n+2}.\end{aligned}\right.$$

Therefore, all the $\frac{n+1}{2}$ unknown quantities can be determined.

Thus, when $n=3$,

$$c=0{\cdot}430\,l^2,\quad x_1=0{\cdot}775\,l;$$

when $n=5$,

$$c=0{\cdot}239\,l^2,\quad x_1=0{\cdot}500\,l,\quad x_2=0{\cdot}892\,l;$$

or $c=0{\cdot}324\,l^2,\quad x_1=0{\cdot}180\,l,\quad x_2=0{\cdot}849\,l.$

When $n=7$, we have six sets of values for c, x_1, x_2 and x_3.

(To find these values, I have proceeded in the following manner. The simultaneous equations to be solved in this case are:

$$(a)\cdots\cdots\left\{\begin{aligned} &x_1+x_2+x_3=\frac{l^3}{3c},\\ &x_1^3+x_2^3+x_3^3=\frac{l^5}{5c},\\ &x_1^5+x_2^5+x_3^5=\frac{l^7}{7c},\\ &x_1^7+x_2^7+x_3^7=\frac{l^9}{9c}.\end{aligned}\right.$$

Put $c=al^2,\ \frac{x_1}{l}=x,\ \frac{x_2}{l}=y,\ \frac{x_3}{l}=z.$

Then the above equations become

$$(a')\cdots\cdots\left\{\begin{aligned} &x+y+z=\frac{1}{3a},\\ &x^3+y^3+z^3=\frac{1}{5a},\\ &x^5+y^5+z^5=\frac{1}{7a},\\ &x^7+y^7+z^7=\frac{1}{9a}.\end{aligned}\right.$$

From the first three equations we have

$$(b)\qquad \beta=\frac{1}{3a\alpha}\left(\alpha^2-\frac{1}{3a}\alpha+\frac{1}{27a^2}-\frac{1}{5}\right)$$

$$(c)\qquad 3a\alpha^3\left(1-\frac{5}{27a^2}\right)-2\alpha^2\left(1-\frac{5}{27a^2}\right)+\alpha\left(\frac{2}{3a}-\frac{9}{7}a-\frac{7}{81a^3}\right)+\frac{1}{5}\left(1-\frac{5}{27a^2}\right)^2=0;$$

where $\alpha=y+z,\ \beta=yz.$

Also from the first and the last equations of (a'),

$$(d)\qquad a^3\alpha^3\left(\frac{8}{27}-a^2-\frac{27}{7}a^4\right)+15a^2\alpha^2\left(\frac{32}{81\times5}-\frac{2}{3}a^2+\frac{54}{35}a^4\right)+\frac{5}{7}a\alpha\left(\frac{47}{729}-\frac{49}{27\times5}a^2-\frac{7}{25}a^4+3a^6\right)-\frac{1}{3}\left(\frac{8}{3^6}-\frac{13}{27\times5}a^2+\frac{2}{35}a^4+\frac{27}{35}a^6\right)=0,$$

by virtue of the relation (c).

Therefore we have to find a and α from equations (c) and (d). These are cubic equations in α, so we can find the general expression for α in terms of a, but as the expression is tolerably complex in this case, I had rather resource to graphic solution, and have obtained the following curves for (c) and (d), (Fig. 2).

Fortunately enough, the solutions are all real and the first approximation to these solutions are:

(i.)	$a=0{\cdot}47$,	$a=0{\cdot}22$;
(ii.)	$a=0{\cdot}48$,	$a=1{\cdot}32$;
(iii.)	$a=0{\cdot}53$,	$a=0{\cdot}12$;
(iv.)	$a=0{\cdot}66$,	$a=-0{\cdot}26$;
(v.)	$a=0{\cdot}67$,	$a=0{\cdot}13$;
(vi.)	$a=2{\cdot}00$,	$a=0{\cdot}03$.

Consequently we have,

when $n=7$,

(i.)	$c=0{\cdot}47l^2$,	$x_1=-0{\cdot}38l$,	$x_2=\ 0{\cdot}49l$,	$x_3=0{\cdot}60l$;
(ii.)	$c=0{\cdot}48l^2$,	$x_1=\ 0{\cdot}52l$,	$x_2=-0{\cdot}62l$,	$x_3=0{\cdot}80l$;
(iii.)	$c=0{\cdot}53l^2$,	$x_1=\ 0{\cdot}51l$,	$x_2=-0{\cdot}76l$,	$x_3=0{\cdot}88l$;
(iv.)	$c=0{\cdot}66l^2$,	$x_1=\ 0{\cdot}29l$,	$x_2=-0{\cdot}55l$,	$x_3=0{\cdot}77l$;
(v.)	$c=0{\cdot}67l^2$,	$x_1=\ 0{\cdot}37l$,	$x_2=-0{\cdot}73l$,	$x_3=0{\cdot}87l$;
(vi.)	$c=2{\cdot}00l^2$,	$x_1=\ 0{\cdot}14l$,	$x_2=-1{\cdot}02l$,	$x_3=1{\cdot}05l$.

This result is only a first approximation and the last figures may perhaps not be correct.

I am now calculating these values for the second approximation and believe in a short time will be able to give more precise values for these quantities.)

ii) Consider the case when n is an even integer.

Let

$$M=c\left\{y_1+y_2+\cdots\cdots+y_{\frac{n}{2}}-(y_{-1}+y_{-2}+\cdots\cdots+y_{-\frac{n}{2}})\right\},$$

as before.

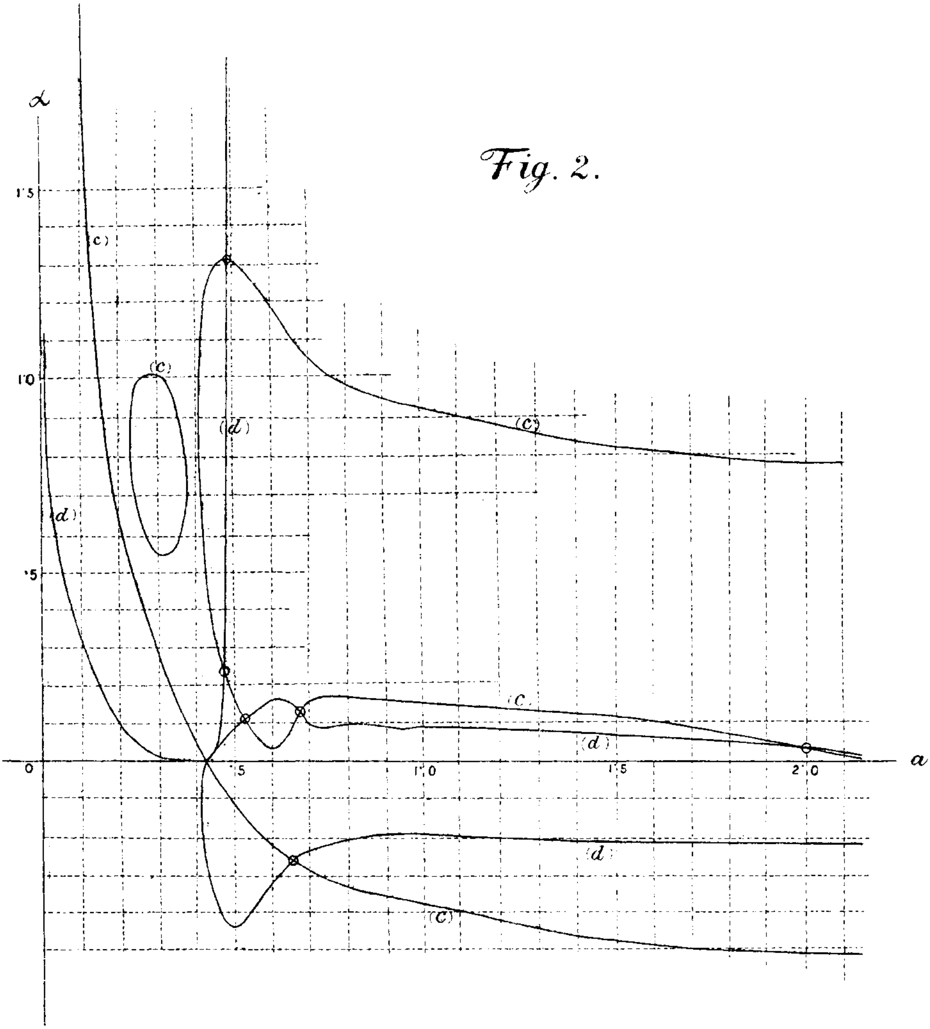
$\therefore$

(4) $$M=2c\left\{a_1(x_1+x_2+\cdots\cdots+x_{\frac{n}{2}})+a_3(x_1^3+x_2^3+\cdots\cdots+x_{\frac{n}{2}}^3)+\cdots\cdots\cdots\cdots+a_{n-1}(x_1^{n-1}+x_2^{n-1}+\cdots\cdots+x_{\frac{n}{2}}^{n-1})\right\}.$$

Comparing (1) and (4), we get the following $\frac{n}{2}$ equations:

(5)$\cdots\cdots\cdots\cdots$ $$\begin{cases} c(x_1+x_2+\cdots\cdots\cdots\cdots+x_{\frac{n}{2}})=\dfrac{l^3}{3}, \\ c(x_1^3+x_2^3+\cdots\cdots\cdots\cdots+x_{\frac{n}{2}}^3)=\dfrac{l^5}{5}, \\ \cdots\cdots\cdots\cdots\cdots\cdots=\cdots\cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots=\cdots\cdots, \\ c(x_1^{n-1}+x_2^{n-1}+\cdots+x_{\frac{n}{2}}^{n-1})=\dfrac{l^{n+1}}{n+1}. \end{cases}$$

Fig. 2.



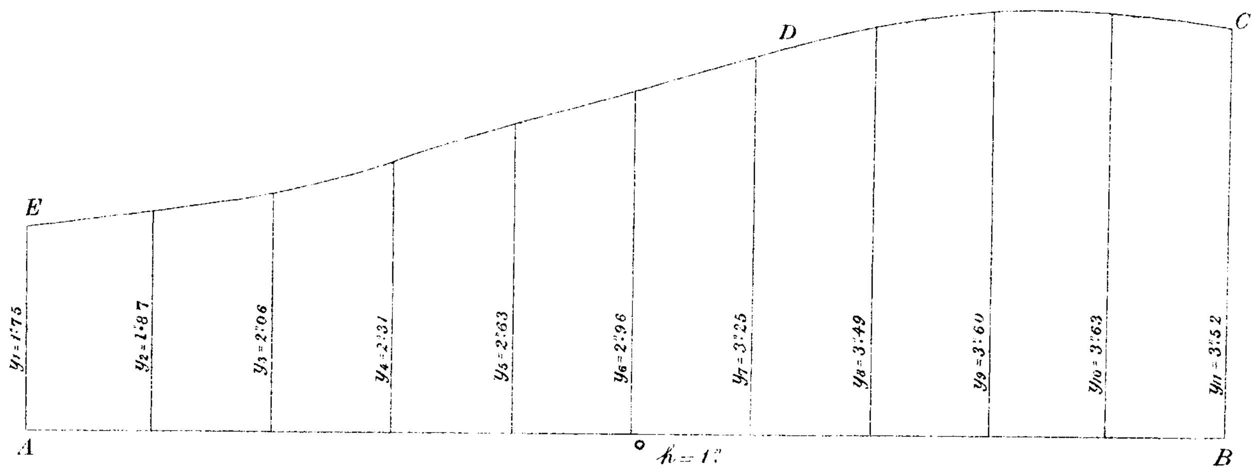

E
D
C
y1 = 1"75
y2 = 1"87
y3 = 2"06
y4 = 2"31
y5 = 2"63
y6 = 2"96
y7 = 3"25
y8 = 3"49
y9 = 3"60
y10 = 3"63
y11 = 3"52
A
O
h = 1"
B

Fig. 3.

Therefore we can give to one of the unknown quantities an arbitrary value whatever we please.

Put $c=l^2$.

Thus, when $n=2$,

$$c=l^2,\quad x_1=\ \ 0{\cdot}333l;$$

when $n=4$,

$$c=l^2,\quad x_1=-0{\cdot}270l,$$
$$x_2=\ \ 0{\cdot}603l;$$

when $n=6$,

$$c=l^2,\quad x_1=\ \ 0{\cdot}185l,$$
$$x_2=-0{\cdot}584l,$$
$$x_3=\ \ 0{\cdot}732l.$$

2. As an example, take the area ABCDE (Fig. 3), and let us compare the results obtained by the above rules with that by Simpson's first rule with eleven ordinates. The use of Simpson's second rule for finding the moment is not so exactly correct as we suppose, except in certain special cases, but the first rule is always correct on the assumption that the curve is made up of portions of parabolas with vertical transverse axes, as I will show in the next chapter.

The moment calculated by the usual formula

$$M=\frac{h^2}{3}\Big\{y_1\times(-5)+4y_2\times(-4)+\cdots\cdots+4y_6\times 0+2y_7\times 1+\cdots\cdots\cdots$$
$$\cdots\cdots\cdots+4y_{10}\times 4+y_{11}\times 5\Big\},$$

where y_1, y_2,........, y_{11} are the lengths of equidistant ordinates and h the horizontal interval, is 18·98 inch units.

The moments calculated by the above rules are:

when	$n=2$,	M = 24·5	inch	units;
"	$n=3$,	" = 18·81	"	" ;
"	$n=4$,	" = 18·00	"	" ;
"	$n=5$,	" = 18·94	"	" ,
		or 18·87	"	" ;
"	$n=6$,	" = 18·00	"	" .

Hence we see for this particular area, remembering that the last one or even two figures in the results may not be correct owing to the error of appreciation and of reading, that the results are practically the same with that by Simpson's rule with eleven ordinates when n is an odd integer. This may be easily judged to be so as the form of the curve in this special example resembles very much a portion of a cubic parabola with vertical transverse axis.

More extended examples are necessary to test the relative accuracies of the above rules.

These rules for calculating M is especially convenient when finding the moment of the volumes of displacement for successive water lines about the midship section.

For example, when $n=5$, if 1, 2, -1, -2 of Fig. 4, be the corresponding sections of a ship at x_1, x_2, x_{-1}, and x_{-2} respectively, we have only to trace successively the peripheries of the areas by the pointer of a planimeter as indicated by arrows; namely, those of the fore body in one direction and the aft body in opposite direction.

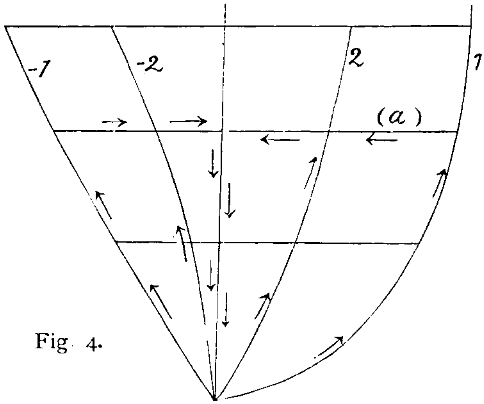


Fig. 4.

The difference of the initial and final readings multiplied by a constant factor gives at once the moment required corresponding to water line (a). Similarly, the difference of the initial and final readings corresponding to any other water line multiplied by the same constant factor gives the moment corresponding to that water line.

When some of x_1, x_2,............are negative, it is better to write all the positive sections on one side of the centre line and the negative sections on the other to avoid a chance for mistake.

3. The moment of inertia ($\mathfrak{I}$) of the area about y axis (Fig. 1) is given by

$$\mathfrak{I}=\int_{-l}^{l} yx^2dx$$

$$=\int_{-l}^{l}(a_0x^2+a_1x^3+\cdots\cdots+a_{n-1}x^{n+1}+a_nx^{n+2})dx.$$

$\therefore$

(6) $$\mathfrak{I}=2l^3\left(\frac{a_{n-1}}{n+2}l^{n-1}+\frac{a_{n-3}}{n}l^{n-3}+\cdots\cdots+\frac{a_2l^2}{5}+\frac{a_0}{3}\right),\ n\ \text{odd:}$$

$$=2l^3\left(\frac{a_n}{n+3}l^n+\frac{a_{n-2}}{n+1}l^{n-2}+\cdots\cdots+\frac{a_2l^2}{5}+\frac{a_0}{3}\right),\ n\ \text{even.}$$

i) When n is an odd integer,
let

$$(7)\qquad \mathfrak{J}= c\left\{y_0+(y_1+y_{-1})+(y_2+y_{-2})+\cdots\cdots+(y_{\frac{n-1}{2}}+y_{-\frac{n-1}{2}})\right\}$$
$$=2c\left\{\tfrac{n}{2}a_0+a_2(x_1^2+x_2^2+\cdots\cdots+x^2_{\frac{n-1}{2}})+\cdots\cdots\cdots\cdots\right.$$
$$\left.\cdots\cdots+a_{n-1}(x_1^{n-1}+x_2^{n-1}+\cdots\cdots+x_{\frac{n-1}{2}}^{n-1})\right\}$$

Comparing the coefficients of a_0, $a_2,\cdots\cdots,a_{n-1}$ in (6) and (7), we have

$$(8)\cdots\cdots\cdots\cdots\left\{\begin{array}{l} c=\dfrac{2}{3n}l^3, \\ x_1^2+x_2^2+\cdots\cdots+x^2_{\frac{n-1}{2}}=\dfrac{3nl^2}{2\times 5}, \\ x_1^4+x_2^4+\cdots\cdots+x^4_{\frac{n-1}{2}}=\dfrac{3nl^4}{2\times 7}, \\ \cdots\cdots\cdots\cdots\cdots\cdots=\cdots\cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots=\cdots\cdots, \\ x_1^{n-1}+x_2^{n-1}+\cdots\cdots+x_{\frac{n-1}{2}}^{n-1}=\dfrac{3nl^{n-1}}{2\times(n+2)}. \end{array}\right.$$

When $n=3$,

$$c=\tfrac{2}{9}l^3,\qquad x_1=0.949l.$$

When $n=5$ or 7, the solutions are unreal.

ii) When n is an even integer,
let

$$(9)\qquad \mathfrak{J}= c\left\{(y_1+y_{-1})+(y_2+y_{-2})+\cdots\cdots+(y_{\frac{n}{2}}+y_{-\frac{n}{2}})\right\}$$
$$=2c\left\{\tfrac{n}{2}a_0+a_2(x_1^2+x_2^2+\cdots\cdots+x^2_{\frac{n}{2}})+\cdots\cdots\cdots\cdots\cdots\cdots\right.$$
$$\cdots\cdots+a_{n-2}(x_1^{n-2}+x_2^{n-2}+\cdots\cdots+x_{\frac{n}{2}}^{n-2})$$
$$\left.+a_n(x_1^n+x_2^n+\cdots\cdots+x_{\frac{n}{2}}^n)\right\}.$$

Comparing the coefficients in (6) and (9), we get

$$(10)\cdots\cdots\cdots\cdots\left\{\begin{array}{l} c=\dfrac{2}{3n}l^3, \\ x_1^2+x_2^2+\cdots\cdots+x^2_{\frac{n}{2}}=\dfrac{3nl^2}{2\times 5}, \\ x_1^4+x_2^4+\cdots\cdots+x^4_{\frac{n}{2}}=\dfrac{3nl^4}{2\times 7}, \\ \cdots\cdots\cdots\cdots\cdots\cdots=\cdots\cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots=\cdots\cdots, \\ x_1^n+x_2^n+\cdots\cdots+x_{\frac{n}{2}}^n=\dfrac{3nl^n}{2\times(n+3)}. \end{array}\right.$$

When $n=2$,

$$c=\tfrac{1}{3}l^3, \qquad x_1=0\cdot775l;$$

when $n=4$,

$$c=\tfrac{1}{6}l^3, \qquad x_1=0\cdot581l,$$
$$x_2=0\cdot928l;$$

when $n=6$,

$$c=\tfrac{1}{9}l^3, \qquad x_1=0\cdot500l,$$
$$x_2=0\cdot815l,$$
$$x_3=0\cdot941l.$$

I think this method is generally more convenient and more accurate than Simpson's rules, and will show this fact with actual examples after I have calculated the values of c_1 x_1, x_2,·······for $n>7$.

4. The moment of inertia I of the area about x axis (Fig. 1) is given by

$$I=\int_{-l}^{l}\tfrac{1}{3}y^3dx$$
$$=\tfrac{1}{3}\int_{-l}^{l}\{a_n^3x^{3n}+3a_n^2a_{n-1}x^{3n-1}+3a_n(a_na_{n-2}+a_{n-1}^2)x^{3n-2}+\cdots\cdots\cdots$$
$$\cdots\cdots\cdots+3a_0(a_0a_2+a_1^2)x^2+3a_0^2a_1x+a_0^3\}\,dx.$$

∴

(11) $$I=\tfrac{2}{3}l\left\{\frac{3a_n^2a_{n-1}}{3n}l^{3n-1}+\cdots\cdots+\frac{3a_0(a_0a_2+a_1^2)l^2}{3}+a_0^3\right\},\ n \text{ odd};$$

$$=\tfrac{2}{3}l\left\{\frac{a_n^3}{3n+1}l^{3n}+\cdots\cdots+\frac{3a_0(a_0a_2+a_1^2)l^2}{3}+a_0^3\right\},\ n \text{ even}.$$

i) Consider the case when n is an odd integer.

Let

(12) $$I=c\left\{y_0^3+(y_1^3+y_{-1}^3)+(y_2^3+y_{-2}^3)+\cdots\cdots+(y_{\frac{3n-1}{2}}^3+y_{-\frac{3n-1}{2}}^3)\right\}$$

$$=2c\left\{\frac{3n}{2}a_0^3+3a_0(a_0a_2+a_1^2)(x_1^2+x_2^2+\cdots\cdots+x_{\frac{3n-1}{2}}^2)+\cdots\cdots\right.$$
$$\left.\cdots\cdots+3a_n^2a_{n-1}(x_1^{3n-1}+x_2^{3n-1}+\cdots\cdots+x_{\frac{3n-1}{2}}^{3n-1})\right\}.$$

Comparing the coefficients of (11) and (12), we have

$$(13)\cdots\cdots\begin{cases} c=\dfrac{2l}{9n}, \\ x_1^2+x_2^2+\cdots\cdots+x_{\frac{3n-1}{2}}^2=\dfrac{3nl^2}{2\times 3}, \\ x_1^4+x_2^4+\cdots\cdots+x_{\frac{4n-1}{3}}^4=\dfrac{3nl^4}{2\times 5}, \\ \cdots\cdots\cdots\cdots=\cdots\cdots, \\ \cdots\cdots\cdots\cdots=\cdots\cdots, \\ x_1^{3n-1}+x_2^{3n-1}+\cdots\cdots+x_{\frac{3n-1}{2}}^{3n-1}=\dfrac{3nl^{3n-1}}{2\times 3n}. \end{cases}$$

When $n=3$, (13) becomes

$$(13')\cdots\cdots\begin{cases} c=\dfrac{2l}{27}, \\ x_1^2+x_2^2+x_3^2+x_4^2=\dfrac{3l^2}{2}, \\ x_1^4+x_2^4+x_3^4+x_4^4=\dfrac{9l^4}{10}, \\ x_1^6+x_2^6+x_3^6+x_4^6=\dfrac{9l^6}{14}, \\ x_1^8+x_2^8+x_3^8+x_4^8=\dfrac{9l^8}{18}. \end{cases}$$

The solution of (13′) is unreal.

ii) Consider the case when n is an even integer.

Let

$$(14)\qquad \mathrm{I}=c\left\{(y_1^3+y_{-1}^3)+(y_2^3+y_{-2}^3)+\cdots\cdots+(y_{\frac{3n}{2}}^3+y_{-\frac{3n}{2}}^3)\right\}$$

$$=2c\left\{\frac{3n}{2}a_0^3+3a_0(a_0a_2+a_1^2)(x_1^2+x_2^2+\cdots\cdots+x_{\frac{3n}{2}}^2)+\cdots\cdots\cdots\right.$$

$$\left.\cdots\cdots+a_n^3(x_1^{3n}+x_2^{3n}+\cdots\cdots+x_{\frac{3n}{2}}^{3n})\right\}.$$

Comparing (11) and (14),

$$(15)\cdots\cdots\begin{cases} c=\dfrac{2l}{9n}, \\ x_1^2+x_2^2+\cdots\cdots+x_{\frac{3n}{2}}^2=\dfrac{3nl^2}{2\times 3}, \\ x_1^4+x_2^4+\cdots\cdots+x_{\frac{3n}{2}}^4=\dfrac{3nl^4}{2\times 5}, \\ \cdots\cdots\cdots\cdots=\cdots\cdots, \\ \cdots\cdots\cdots\cdots=\cdots\cdots, \\ x_1^{3n}+x_2^{3n}+\cdots\cdots+x_{\frac{3n}{2}}^{3n}=\dfrac{3nl^{3n}}{2\times(3n+1)}. \end{cases}$$

Thus, when $n=2$, we have

$$c=\frac{l}{9}, \qquad x_1=0{\cdot}272l,$$
$$x_2=0{\cdot}419l,$$
$$x_3=0{\cdot}866l.$$

The number of abscissæ to be measured increases rapidly with n, but the accuracy of the result is surely very much superior to that by Simpson's rule, as the latter will sometimes give rise to results very different from true values for I, as we will see in the next chapter.

I hope I can give in a near future the complete results of calculations corresponding to higher values of n useful for several rules explained in this paper

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II. *Analysis of Simpson's rules for the moment and moment of inertia of a plane figure about an axis.*

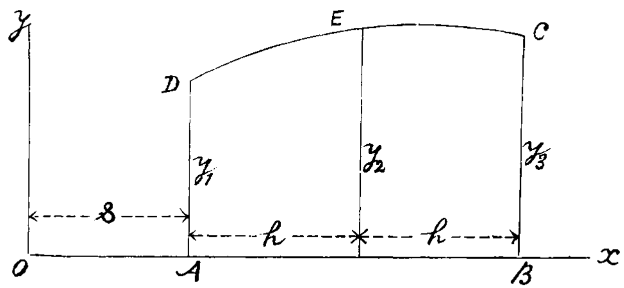

Fig. 5.

5. Take ABCED (Fig. 5) to be a plane area we have to consider. Take the y axis at a distance s from the end ordinate y_1, and assume the curve DEC to be a part of

$$y=ax^2+bx+c.$$

The moment about y axis is given by

$$(1) \qquad \mathfrak{M}=\int_s^{s+2h} yxdx=\left[\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}\right]_s^{s+2h}$$

But

$$y_1=as^2+bs+c,$$
$$y_2=a(s+h)^2+b(s+h)+c,$$
$$y_3=a(s+2h)^2+b(s+2h)+c.$$

Multiplying to both members of these equations s, $4(s+h)$, $s+2h$ respectively and substituting the results in (1), we have finally

(2) $\mathfrak{M}=\frac{h}{3}\{sy_1+4(s+h)y_2+(s+2h)y_3\}$,

the usual formula for the moment.

Therefore this formula is right in all cases on the assumption that the curve DEC is a part of $y=ax^2+bx+c$.

6. Nextly, take Fig. 6, and assume the curve DEC to be a part of

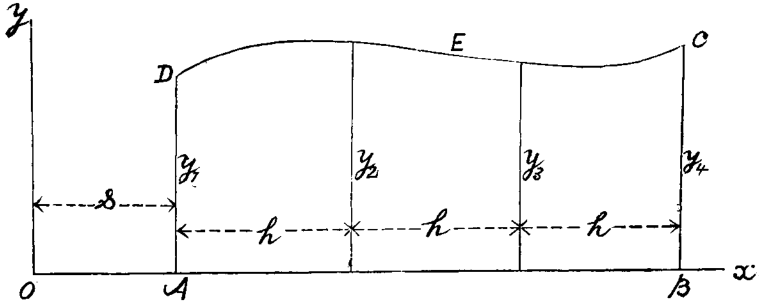

Fig. 6.

$$y=ax^3+bx^2+cx+f.$$

The moment about y axis at a distance s from y_1 is given by

$$M=\int_s^{s+3h} yxdx=\left[\frac{ax^5}{5}+\frac{bx^4}{4}+\frac{cx^3}{3}+\frac{fx^2}{2}\right]_s^{s+3h}.$$

Put $s=mh$.

$\therefore$

(3) $$M=h^2\left\{\frac{a}{5}(\overline{m+3}^5-m^5)h^3+\frac{b}{4}(\overline{m+3}^4-m^4)h^2 + \frac{c}{3}(\overline{m+3}^3-m^3)h+\frac{f}{2}(\overline{m+3}^2-m^2)\right\}$$

But

$$y_1=a\,\overline{mh}^3+b\,\overline{mh}^2+c\,\overline{mh}+f,$$
$$y_2=a\,\overline{m+1}^3h^3+b\,\overline{m+1}^2h^2+c\,\overline{m+1}\,h+f,$$
$$y_3=a\,\overline{m+2}^3h^3+b\,\overline{m+2}^2h^2+c\,\overline{m+2}\,h+f,$$
$$y_4=a\,\overline{m+3}^3h^3+b\,\overline{m+3}^2h^2+c\,\overline{m+3}\,h+f.$$

Multiply to both members of these equations the undetermined multipliers λ, μ, ν and ξ respectively and add them together member by member.

$\therefore$

$$(4)\qquad \lambda y_1+\mu y_2+\nu y_3+\xi y_4=a(m^3\lambda+\overline{m+1}^{3}\mu+\overline{m+2}^{3}\nu+\overline{m+3}^{3}\xi)h^3$$
$$+b(m^2\lambda+\overline{m+1}^{2}\mu+\overline{m+2}^{2}\nu+\overline{m+3}^{2}\xi)h^2$$
$$+c(m\lambda+\overline{m+1}\,\mu+\overline{m+2}\,\nu+\overline{m+3}\,\xi)h$$
$$+f(\lambda+\mu+\nu+\xi).$$

Comparing the coefficients of a, b, c and f in (3) and (4), we have

$$\begin{cases} \frac{1}{5}(\overline{m+3}^{5}-m^5)=m^3\lambda+\overline{m+1}^{3}\mu+\overline{m+2}^{3}\nu+\overline{m+3}^{3}\xi, \\ \frac{1}{4}(\overline{m+3}^{4}-m^4)=m^2\lambda+\overline{m+1}^{2}\mu+\overline{m+2}^{2}\nu+\overline{m+3}^{2}\xi, \\ \frac{1}{3}(\overline{m+3}^{3}-m^3)=m\lambda+\overline{m+1}\,\mu+\overline{m+2}\,\nu+\overline{m+3}\,\xi, \\ \frac{1}{2}(\overline{m+3}^{2}-m^2)=\lambda+\mu+\nu+\xi. \end{cases}$$

$\therefore$

$$\lambda=\frac{\begin{vmatrix} \frac{1}{5}(\overline{m+3}^{5}-m^5) & \overline{m+1}^{3} & \overline{m+2}^{3} & \overline{m+3}^{3} \\ \frac{1}{4}(\overline{m+3}^{4}-m^4) & \overline{m+1}^{2} & \overline{m+2}^{2} & \overline{m+3}^{2} \\ \frac{1}{3}(\overline{m+3}^{3}-m^3) & \overline{m+1} & \overline{m+2} & \overline{m+3} \\ \frac{1}{2}(\overline{m+3}^{2}-m^2) & 1 & 1 & 1 \end{vmatrix}}{\Delta},$$

$$\mu=\frac{\begin{vmatrix} m^3 & \frac{1}{5}(\overline{m+3}^{5}-m^5) & \overline{m+2}^{3} & \overline{m+3}^{3} \\ m^2 & \frac{1}{4}(\overline{m+3}^{4}-m^4) & \overline{m+2}^{2} & \overline{m+3}^{2} \\ m & \frac{1}{3}(\overline{m+3}^{3}-m^3) & \overline{m+2} & \overline{m+3} \\ 1 & \frac{1}{2}(\overline{m+3}^{2}-m^2) & 1 & 1 \end{vmatrix}}{\Delta},$$

$$\nu=\frac{\begin{vmatrix} m^3 & \overline{m+1}^{3} & \frac{1}{5}(\overline{m+3}^{5}-m^5) & \overline{m+3}^{3} \\ m^2 & \overline{m+1}^{2} & \frac{1}{4}(\overline{m+3}^{4}-m^4) & \overline{m+3}^{2} \\ m & \overline{m+1} & \frac{1}{3}(\overline{m+3}^{3}-m^3) & \overline{m+3} \\ 1 & 1 & \frac{1}{2}(\overline{m+3}^{2}-m^2) & 1 \end{vmatrix}}{\Delta},$$

$$\xi=\frac{\begin{vmatrix} m^3 & \overline{m+1}^{3} & \overline{m+2}^{3} & \frac{1}{5}(\overline{m+3}^{5}-m^5) \\ m^2 & \overline{m+1}^{2} & \overline{m+2}^{2} & \frac{1}{4}(\overline{m+3}^{4}-m^4) \\ m & m+1 & m+2 & \frac{1}{3}(\overline{m+3}^{3}-m^3) \\ 1 & 1 & 1 & \frac{1}{2}(\overline{m+3}^{2}-m^2) \end{vmatrix}}{\Delta};$$

where

$$\Delta \equiv \begin{vmatrix} m^3 & \overline{m+1}^3 & \overline{m+2}^3 & \overline{m+3}^3 \\ m^2 & \overline{m+1}^2 & \overline{m+2}^2 & \overline{m+3}^2 \\ m & m+1 & m+2 & m+3 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

Or

$$(5) \cdots\cdots \begin{cases} \lambda = \dfrac{3(5m+2)}{40}, \\ \mu = \dfrac{9(5m+3)}{40}, \\ \nu = \dfrac{9(5m+12)}{40}, \\ \xi = \dfrac{3(5m+13)}{40}. \end{cases}$$

$\therefore$

$$(6) \qquad M = \frac{3h^2}{40}\Big\{(5m+2)y_1 + 3(5m+3)y_2 + 3(5m+12)y_3 + (5m+13)y_4\Big\}.$$

The usual formula for the moment is

$$(7) \qquad M' = \frac{3h^2}{8}\Big\{my_1 + 3(m+1)y_2 + 3(m+2)y_3 + (m+3)y_4\Big\}.$$

Let us see the difference between these two formulæ (6) and (7).

$$(8) \qquad M - M' = \frac{3h^2}{40}\Big\{2y_1 - 6y_2 + 6y_3 - 2y_4\Big\}$$

$$= \frac{3h^2}{20}\Big\{(y_1 - y_4) - 3(y_2 - y_3)\Big\}.$$

Therefore the formula (7) does not always give the correct value of M and the error of that formula is given by (8) on the assumption that the curve DEC is a part of

$$y = ax^3 + bx^2 + cx + f.$$

For example, put

$$m=0,\ h=1,\ y_1=5,\ y_2=4.5,\ y_3=4,\ y_4=0.$$

The moments by (6) and (7) are respectively 14·5875 and 14·0625 units and the difference between these is 0·525 units or about $\frac{1}{28}$ of the true moment.

As another extreme example, take

$$m=0,\ h=1,\ y_1=5,\ y_2=0,\ y_3=5,\ y_4=0.$$

The moments by (6) and (7) are respectively 14·25 and 11·25 units and the difference is 3 units or about $\frac{1}{5}$ of the true moment.

7. The moment of the area between ordinates y_1 and y_2 about y axis (Fig. 5) is

(9) $$\mathfrak{M}_{12}=\int_s^{s+h} yx\,dx$$

$$=\left[\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}\right]_{mh}^{(m+1)h},$$

by putting $s=mh$.

But,

$$y_1=a\,\overline{mh}^2+bmh+c,$$
$$y_2=a\,\overline{m+1}^2h^2+b\,\overline{m+1}\,h+c,$$
$$y_3=a\,\overline{m+2}^2h^2+b\,\overline{m+2}\,h+c.$$

Therefore, multiplying the undetermined multipliers λ, μ, ν to the above equations and adding them together member by member, we get

(10) $$\lambda y_1+\mu y_2+\nu y_3=ah^2(\lambda m^2+\mu\,\overline{m+1}^2+\nu\,\overline{m+2}^2)$$
$$+bh\,(\lambda m+\mu\,\overline{m+1}+\nu\,\overline{m+2})$$
$$+\;c\,(\lambda+\mu+\nu).$$

Comparing (9) and (10), as before,

(11) $$\begin{cases} 3(4m^3+6m^2+4m+1)=\lambda m^2+\mu\,\overline{m+1}^2+\nu\,\overline{m+2}^2, \\ 4(3m^2+3m+1)=\lambda m+\mu\,\overline{m+1}+\nu\,\overline{m+2}, \\ 6(2m+1)=\lambda+\mu+\nu. \end{cases}$$

$\therefore$

$$\lambda=\frac{\begin{vmatrix} 3(4m^3+6m^2+4m+1) & \overline{m+1}^2 & \overline{m+2}^2 \\ 4(3m^2+3m+1) & m+1 & m+2 \\ 6(2m+1) & 1 & 1 \end{vmatrix}}{\Delta},$$

$$\mu=\frac{\begin{vmatrix} m^2 & 3(4m^3+6m^2+4m+1) & \overline{m+2}^2 \\ m & 4(3m^2+3m+1) & m+2 \\ 1 & 6(2m+1) & 1 \end{vmatrix}}{\Delta},$$

$$\nu=\frac{\begin{vmatrix} m^2 & \overline{m+1}^2 & 3(4m^3+6m^2+4m+1) \\ m & m+1 & 4(3m^2+3m+1) \\ 1 & 1 & 6(2m+1) \end{vmatrix}}{\Delta};$$

where

$$\Delta \equiv \begin{vmatrix} m^2 & \overline{m+1}^2 & \overline{m+2}^2 \\ m & m+1 & m+2 \\ 1 & 1 & 1 \end{vmatrix}$$

∴

$$(12) \cdots\cdots\cdots\cdots \begin{cases} \lambda = \frac{10m+3}{2}, \\ \mu = \frac{8m+5}{1}, \\ \nu = \frac{-2m-1}{2}. \end{cases}$$

Therefore,

$$(13) \qquad \mathfrak{M}_{12} = \frac{h^2}{24}\left\{ (10m+3)y_1 + 2(8m+5)y_2 - (2m+1)y_3 \right\}$$

Similarly, the moment $\mathfrak{M}_{23}$ of the area between y_2 and y_3 about y axis is

$$(14) \qquad \mathfrak{M}_{23} = \frac{h^2}{24}\left\{ -(2m+3)y_1 + 2(8m+11)y_2 + (10m+17)y_3 \right\}.$$

The sum of (13) and (14) is

$$\mathfrak{M}_{12} + \mathfrak{M}_{23} = \frac{h^2}{24}\left\{ 8my_1 + 2(16m+16)y_2 + (8m+16)y_3 \right\}$$

$$= \frac{h^2}{3}\left\{ my_1 + 4(m+1)y_2 + (m+2)y_3 \right\}.$$

Thus, we have again obtained the formula (2), as it ought to be so.

8. The moment of the area about x axis (Fig. 5) is

$$M_x = \int_{-h}^{h} \tfrac{1}{2} y^2 dx,$$

by taking the origin at the middle of the base. The generality of the problem is not lost by this assumption.

∴

$$(15) \qquad M_x = \tfrac{1}{2}\int_{-h}^{h} \{a^2x^4 + 2abx^3 + (b^2+2ac)x^2 + 2bcx + c^2\} dx$$

$$= \tfrac{1}{2}\left[\frac{a^2x^5}{5} + \frac{2abx^4}{4} + \frac{(b^2+2ac)x^3}{4} + \frac{2bcx^2}{2} + c^2x \right]_{-h}$$

$$= h\left\{ \frac{a^2}{5}h^4 + \frac{(b^2+2ac)}{3}h^2 + c^2 \right\}.$$

But,

$$y_1 = ah^2 - bh + c,$$

$y_2 = c,$

$y_3 = ah^2 + bh + c.$

$$\therefore \quad a = \frac{y_1 + y_3 - 2y_2}{2h^2}, \qquad b = \frac{y_3 - y_1}{2h}, \qquad c = y_2.$$

Substituting these values in (15), we have

$$(16) \qquad M_x = \frac{2h}{15}\left[y_1^2 + 4y_2^2 + y_3^2 + y_1 y_2 + y_2 y_3 - \tfrac{1}{2} y_3 y_1 \right]$$

The formula in common use for the moment M'_x is

$$(17) \qquad M'_x = \frac{h}{6}(y_1^2 + 4y_2^2 + y_3^2).$$

The difference between M_x and M'_x is

$$M'_x - M_x = \frac{h}{30}\left\{ (y_1 + y_3)(y_1 + y_3 - 4y_2) + 4y_2^2 \right\}.$$

To discuss the amount of the error for variable values of y_1, y_2 and y_3, put

$$y_2 = my_1, \quad y_3 = ny_1$$

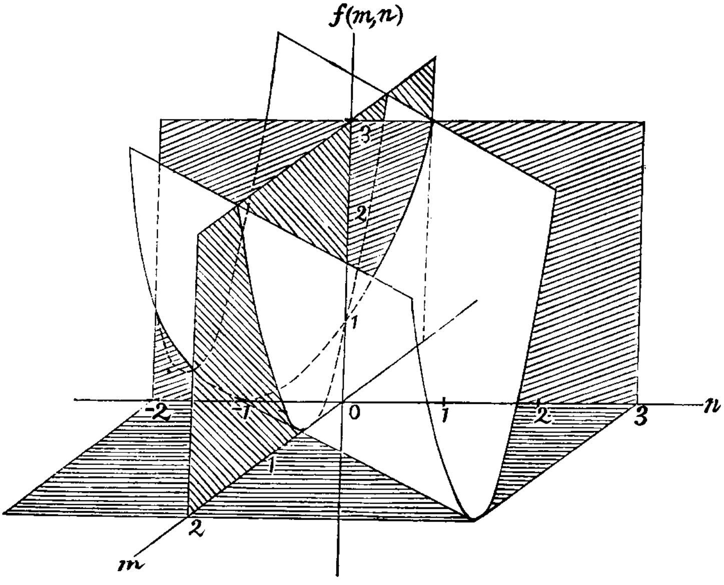

Fig. 7.

and firstly suppose $y_1 \neq 0$.

$\therefore$

$$M'_x - M_x = \frac{hy_1^2}{30}\left\{ (1+n)(1+n-4m)+4m^2 \right\}.$$

Put $\quad f(m,n)=(1+n)(1+n-4m)+4m^2.$

If we take m, n and $f(m,n)$ for rectanguler coordinate axes, the above equation represents a parabolic cylinder (Fig. 7), whose generating lines are parallel to $m\ n$ plane.

$\therefore$ M'_x is always greater than M_x, and only equal to M_x along the straight line $2m-n-1=0$, provided $y_1 \neq 0$.

If $y_1=0$, put

$$y_2=my_3, \quad y_1=ny_3.$$

Then, as the expression $M'_x - M_x$ is symmetrical with respect to y_1 and y_3, we can proceed in exactly the same reasoning, so that the position and the nature of the cylinder remain unaltered.

If $y_1=y_3=0$, $M'_x - M_x$ reduces to

$$M'_x - M_x = \frac{h}{30} \times 4y_2^2,$$

and the error is proportional to y_2^2 for h=const.

9. The moment of inertia of the area about x axis is

$$I= \int_{-h}^{h} \frac{y^3}{3} dx,$$

taking the origin at the middle of the base, (Fig. 5).

$\therefore$

(18) $$I=\frac{1}{3}\int_{-h}^{h}(ax^2+bx+c)^3dx$$

$$=\frac{2}{3}h\left\{\frac{a^3h^6}{7}+\frac{3a(b^2+ac)}{5}h^4+c(ac+b^2)h^2+c^3\right\}.$$

Substituting the values for a, b and c obtained in the preceding paragraph, we have

(19) $$I=\frac{h}{3\times 35\times 2}\left\{13y_1^3+64y_2^3+13y_3^3+4y_2(5\overline{y_1^2+y_3^2}+4y_2\overline{y_1+y_3})\right.$$
$$\left.-y_1y_3(3y_1+3y_3+16y_2)\right\}$$

But the formula for usual calculation for the moment of inertia I′ is

(20) $$I'=\frac{h}{9}(y_1^3+4y_2^3+y_3^3).$$

The difference is

$$I'-I=\frac{h}{9\times 35\times 2}\{y_1^2(31y_1-60y_2+9y_3)+8y_2^2(-6y_1+11y_2-6y_3)$$
$$+y_3^2(9y_1-60y_2+31y_3)+48y_1y_2y_3\}.$$