On the Form of an Arch.

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I have remarked in my paper, "A Note on the Backing of an Arch", that the solution of the form of an arch, with parallel intrados and extrados, whose line of resistance is parallel to them, involves certain difficulties. The present discussion will bring the points of such difficulties clearer in view. I shall try at first to start from the most general case and proceed step by step towards the simpler cases admitting of the solutions available to practical purposes.

If,

$T=$ thrust at any point of the line of resistance making an angle θ with the axis of x in the free surface and parallel to the heads of the arch, having its positive direction towards the left,

$x,y=$ coordinates of the point at which the line through the point under consideration and normal to the intrados or extrados meets the latter, the axis of y being taken through the crown with its positive direction downward.

$t=$ thickness of the arch ring,

$s=$ length of the extradosal curve,

$w=$ unit weight of earth,

$W=$ ditto of arch ring $=\sigma w$ say,

$c=$ ratio of the horizontal pressure of the earth to the vertical,

then the equation of the extrados will be given by

$$1)\begin{cases} d\ (T\cos\theta) = -cwydy \\ d\ (T\sin\theta) = wydx + Wtds - \dfrac{Wt^2}{2}d\theta. \end{cases}$$

The first relation gives

$$T\cos\theta = \frac{w}{2}\left\{b^2 - c(y^2 - a^2)\right\},$$

where b is a constant, $wb^2/2$ is the horizontal crown thrust. and a the surcharge at the crown. Substituting this value of T in the second, we obtain

$$\frac{1}{2}d\left\{(b^2 - \overline{cy^2 - a^2})\frac{dy}{dx}\right\} = ydx + \sigma tds - \frac{\sigma t^2}{2}d\theta,$$

whence putting $dy/dx=p$,

$$\frac{1}{2}\left\{b^2-c(y^2-a^2)\right\}p\frac{dp}{dy}-cyp^2=y+\sigma t\sqrt{1+p^2}-\frac{\sigma t^2}{2}\frac{p}{1+p^2}\frac{dp}{dy},$$

or making $1+p^2=q^2$,

$$2)\quad \left\{b^2-c(y^2-a^2)+\frac{\sigma t^2}{q^2}\right\}\frac{qdq}{dy}=2(1-c+cq^2)y+2\sigma tq.$$

It appears to me to be impossible to arrive at the general solution of this differential equation, except in certain particular cases which I proceed to consider.

Firstly, if $c=1$, since $q\neq 0$, the equation is simplified to

$$\left\{t^2-(y^2-a^2)+\frac{\sigma t^2}{q^2}\right\}\frac{dq}{dy}=2qy+2\sigma t,$$

which may be put in the form

$$\left\{b^2-(y^2-a^2)\right\}aq-2qydy+\sigma t^2\frac{dq}{q^2}-2\sigma tdy=0,$$

whence by integration,

$$(b^2+a^2-y^2)q-\frac{\sigma t^2}{q}-2\sigma ty=2k,$$

where

$$2k=b^2-\sigma t^2-2\sigma ta.$$

Hence

$$(b^2+a^2-y^2)q^2-2(\sigma ty+k)q=\sigma t^2,$$

or, since q is to be positive,

$$q=\frac{\sigma ty+k+\sqrt{(\sigma ty+k^2)+\sigma t^2(b^2+a^2-y^2)}}{b^2+a^2-y^2}$$

But since $q=\sqrt{1+p^2}$ and $dy/dx=p$, we finally obtain

$$3)\quad x=\int_a^y\frac{(b^2+a^2-y^2)dy}{\sqrt{\alpha'+4\sigma tky+\beta'y^2-y^4+2(\sigma ty+k)\sqrt{\gamma'y^2+2\sigma tky+\delta'}}}$$

where

$$\alpha'=2k^2+\sigma(b^2+a^2)t^2-(b^2+a^2)^2$$
$$\beta'=2\sigma^2t^2-\sigma t^2+2(b^2+a^2)$$
$$\gamma'=\sigma^2t^2-\sigma t^2$$
$$\delta'=k^2+(b^2+a^2)t^2$$

This value of x could be determined perhaps most simply by means of graphic calculations; still it is probably too complex for practical purposes.

Returning now to equation 1) we observe that the last term in the second relation plays but a trivial effect upon the nature of the curve. In fact, no material change will be encountered in taking the length of the centre line of the arch ring equal to that of extrados, *i.e.*, in neglecting the above-mentioned term. With this approximation, equation 3) is simplified to

$$4)\qquad x=\int_a^y \frac{(b^2+a^2-y^2)\,dy}{\sqrt{4(\sigma ty+k)^2-(b^2+a^2-y^2)^2}},$$

where

$$2k=b^2-2\sigma ta,$$

and this is evidently an elliptic integral. To transform the expression under the radical of the denominator into a biquadratic form by the substitution

$$y=\frac{\alpha z+\beta}{z+1}$$

and that α and β may be *real* it is necessary to discuss the nature of the roots of the expression. A little investigation will show without difficulty that *the expression admits of four real roots, which, in the order of their magnitudes are*

$$\sigma+h,\ a,\ -2\sigma t-a,\ \sigma t-h,$$

where

$$h=\left|\sqrt{2b^2+(\sigma t-a)^2}\right|,$$

so that we shall have

$$5)\ x=(\alpha-\beta)\int_{\frac{\beta-a}{a-\alpha}}^{\frac{\beta-y}{y-\alpha}} \frac{(b^2+a^2-\alpha^2)y^2+2(b^2+a^2-\alpha\beta)y+b^2+a^2-\beta^2}{(y+1)^2\sqrt{(m+ly^2)(\mu-\lambda y^2)}}\,dy,$$

where

$$\alpha=\frac{\sigma t(a+\sigma t-h)+\sqrt{(a+\sigma t)h(a-\sigma t+h)(a+3\sigma t+h)}}{a+\sigma t+h},$$

$$\beta=\frac{\sigma t(a+\sigma t-h)-\sqrt{(a+\sigma t)h(a-\sigma t+h)(a+3\sigma t+h)}}{a+\sigma t+h},$$

$$l=\alpha(h+\sigma t+a)-a(h+\sigma t)-\alpha^2,$$
$$m=\beta(h+\sigma t+a)-a(h+\sigma t)-\beta^2,$$
$$\lambda=\alpha(h+\sigma t+a)+(h-\sigma t)(2\sigma t+a)+\alpha^2,$$
$$\mu=\beta(h+\sigma t+a)+(h-\sigma t)(2\sigma t+a)+\beta^2.$$

These six constants will be comparatively simply calculated as many of their factors are the same. It will be unnecessary to proceed further to simplify the formula 5), and it can be shewn that the integral depends upon the elliptic integrals of the first, second and third kind.

So far for the case $c=1$. Nextly, if $c=0$ and we neglect the last term in the second relation of equation 1), we shall have

$$\frac{b^2}{2}\frac{qdq}{dy}=y+\sigma tq,$$

whence by putting $q=\xi y$,

$$\frac{b}{2}\xi y d\xi=(1+\sigma t\xi-\frac{b^2}{2}\xi^2)dy.$$

Iutegrating this equation, and observing that $q=1$ when $y=a$, we arrive at

$$6)\quad \frac{y^2+\sigma tyq-\frac{b^2}{2}q^2}{a^2+\sigma ta-\frac{b^2}{2}}=\left\{\frac{(\sqrt{2b^2+\sigma^2t^2}-\sigma t)y+b^2q}{(\sqrt{2b^2+\sigma^2t^2}+\sigma t)y-b^2q}\cdot\frac{(\sqrt{2b^2+\sigma^2t^2}+\sigma t)a-b^2}{(\sqrt{2b^2+\sigma^2t^2}-\sigma t)a+b^2}\right\}^{\frac{\sigma t}{\sqrt{2b^2+\sigma^2t^2}}}.$$

It appears to me to be impossible to proceed to further integration, so that the case does not seem to be of any practical utility. The formula will evidentiy lead to the transformed catenarian curve when $t=0$ or $\sigma=0$, as it should be.