A Note on Skew Arches.

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It is the object of this paper to give certain properties of the heading and coursing surfaces as well as their intersections with the intradosal and extradosal surfaces, with some short remarks on the stability of skew arches.

I. Helicoidal Arches.

§ 1. *General Considerations.*

Under this heading are included all those skew arches, whose heading surface is a helical surface * whose directrix of its generating right line is the line in the plane of parallel springing lines equidistant and parallel to them, the axial length of the helix on the soffit being equal to the projection of the span length on a springing line, and the coursing surface is another helical surface with the same directrix for its generating right line, such that the coursing joints may be normal to the heading joints on the soffit, the heads of the arch being assumed to be parallel.

Taking then the directrix of these surfaces as the axis of z, the line through the centre of the locus of crown and normal to the plane of the parallel springing lines as one of y, and the line normal to the plane of yz as one of x, we have for the equation of the intradosal surface or soffit

1) $$y = \phi(x),$$

and for that of the extradosal surface

2) $$y = \psi(x).$$

Since the equation of the head of the arch on the positive side of z is

$$z + \frac{m}{\lambda}x - \frac{l}{2} = 0$$

where l is the length of the arch, the equations of the intrados on the head are

* It may not be without interest to remark here that a helicoid is superposable on a catenoid, its rectilinear generatrix corresponding to the meridian of the latter and the helices to the parallels. For details in this connetion and many important investigations on the surface, refer to *Darboux— Leçons sur la théorie générale des surfaces et les applications géometriques du calcul infinitesimal*, 1re *Partie, Livre I, Ch. VIII*, and to *Greenhill— Theory of Elliptic Functions*.

$$3)\quad \begin{cases} y = \phi(x) \\ z + \dfrac{m}{\lambda}x - \dfrac{l}{2} = 0. \end{cases}$$

The projections of this curve on the coordinate planes may be easily obtained and discussed. The equations of its development on a plane parallel to xz are, denoting the axis corresponding to x—axis by that of ξ,

$$4)\quad \begin{cases} \xi = \displaystyle\int_0^x \sqrt{1 + \left|\frac{d\phi}{dx}\right|^2}\, dx \\ z = \dfrac{l}{2} - \dfrac{m}{\lambda}x \end{cases}$$

where x is the parameter.

Again, the equation of a helical surface satisfying the above conditions is.

$$z = g + a\, tg^{-1}\frac{y}{x},$$

where g and a are constants. Since the heading joint on the soffit has the axial length equal to the projection of the span length λ on a springing line, we shall have, if

$$m = \lambda\, cos\delta,$$

where δ is the angle of obliquity of the arch,

$$\left| z \right|_{\substack{x = -\frac{\lambda}{2} \\ y = 0}} - \left| z \right|_{\substack{x = +\frac{\lambda}{2} \\ y = 0}} = m,$$

whence

$$a = \frac{m}{\pi},$$

so that we may put the equation of the heading surface in the form

$$5)\quad z = g + \frac{m}{\pi} tg^{-1} \frac{y}{x}.$$

The equations of the heading joints are then

$$6)\quad \begin{cases} y = \phi(x) \\ z = g + \dfrac{m}{\pi} tg^{-1} \dfrac{y}{x}. \end{cases}$$

The projections of this curve on the coordinate planes may also be readily traced, while the equations of its development are

$$7)\qquad \begin{cases} \xi = \displaystyle\int_0^x \sqrt{1 + \left|\frac{d\phi}{dx}\right|^2}\, dx \\ z = g + \dfrac{m}{\pi}\, tg^{-1}\, \dfrac{\phi(x)}{x}, \end{cases}$$

where x is the parameter.

Now since the condition of perpendicularity of two curves

$$f(x \mid y \mid z) = 0, \qquad \phi(x \mid y \mid z) = 0$$

and

$$F(x \mid y \mid z) = 0, \qquad \phi(x \mid y \mid z) = 0$$

is

$$\begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial \phi}{\partial y} \\ \frac{\partial f}{\partial z} & \frac{\partial \phi}{\partial z} \end{vmatrix} \begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial \phi}{\partial y} \\ \frac{\partial F}{\partial z} & \frac{\partial \phi}{\partial z} \end{vmatrix} + \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial \phi}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial \phi}{\partial x} \end{vmatrix} \begin{vmatrix} \frac{\partial F}{\partial z} & \frac{\partial \phi}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial \phi}{\partial x} \end{vmatrix} + \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial \phi}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial \phi}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial \phi}{\partial x} \\ \frac{\partial F}{\partial y} & \frac{\partial \phi}{\partial y} \end{vmatrix} = 0,$$

and since the equation of the coursing surface is of the form

$$8)\qquad z = h - a\ tg^{-1} \cdot \frac{y}{x},$$

we must have

$$\begin{vmatrix} \frac{m}{\pi}\frac{x}{x^2+y^2} & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} \frac{ax}{x^2+y^2} & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ \frac{m}{\pi}\frac{y}{x^2+y^2} & -\frac{d\phi}{dx} \end{vmatrix} \begin{vmatrix} 1 & 0 \\ -\frac{ay}{x^2+y^2} & -\frac{d\phi}{dx} \end{vmatrix}$$

$$+ \begin{vmatrix} \frac{m}{\pi}\frac{y}{x^2+y^2} & -\frac{d\phi}{dx} \\ -\frac{m}{\pi}\frac{x}{x^2+y^2} & 1 \end{vmatrix} \begin{vmatrix} -\frac{ay}{x^2+y^2} & -\frac{d\phi}{dx} \\ \frac{ax}{x^2+y^2} & 1 \end{vmatrix} = 0,$$

whence

$$9)\qquad a = \frac{\pi}{m}\ \frac{\left\{x^2 + \phi^2(x)\right\}^2 \left\{1 + \left|\frac{d\phi}{dx}\right|^2\right\}}{\left\{\phi(x) - x\frac{d\phi}{dx}\right\}^2}$$

in which right-hand member the value of x satisfying the condition

$$\phi(x) = x\ tg\ \frac{\pi\ (h-g)}{a\pi + m}$$

is to be put for x.

The equations of coursing joint, of its projections on the coordinate planes and of its development can be arrived at by exactly the same methods as in the case of heading jóint.

To discuss about the stability of such arches, I shall assume, as usual, the line of resistance to be in a plane parallel to the head of the arch. The angle ϕ between the coursing surface and the head can be shewn to be given by

$$10) \qquad \cos\phi = \frac{\sin\delta - \sigma\sin\theta\cos\delta}{\sqrt{1+\sigma^2}},$$

in which

$$\sigma = a(x^2+y^2)^{-\frac{1}{2}},$$
$$\sin\theta = y(x^2+y^2)^{-\frac{1}{2}}.$$

If therefore $\sigma > tg\,\delta$, then the arch will have the shears of opposite senses along the coursing beds at the crown and the springing lines. That the arch may have such shear not exceeding, in magnitude, that at crown, we must have

$$11) \qquad |\cos\phi|_{\theta=\frac{\pi}{2}} \geqq -|\cos\phi|_{\theta=\theta_0},$$

where $\pi - 2\theta_0$ is the central angle subtended by the springing lines at the axis of z. Only in cases

$$|(\cos\phi)_{\theta=\frac{\pi}{2}}| \leqq |(\cos\phi)_{\theta=0}|,$$

the arch may be full-centered (*i.e.*, the central angle may not be decreased).

Another point to be noticed in connection with the stability of the arch is the angle χ between the head and the soffit, which may endanger the strength of the voussoir below a certain limit, say 60°. This angle is given by

$$\cos\chi = \frac{m\phi'(x)}{\sqrt{\lambda^2+m^2}\sqrt{1+\{\phi'(x)\}^2}},$$

or if we put $\cos\theta = x(x^2+y^2)^{-\frac{1}{2}}$,

$$12) \qquad \cos\chi = \cos\theta\cos\delta.$$

With the limit above mentioned

$$\cos\theta = \tfrac{1}{2}\sec\delta.$$

§ 2. *Application to a Circular Arch.*

Denoting the radius of the intrados by r, the equation of the soffit is

$$13) \qquad y^2 + x^2 - r^2 = 0.$$

The equations of the intrados at the head are

14) $$\begin{cases} y^2 + x^2 - r^2 = 0 \\ z + \dfrac{m}{2r}x - \dfrac{l}{2} = 0, \end{cases}$$

whose projections on the coordinate planes are

15) $$y^2 + x^2 - r^2 = 0$$

16) $$z + \frac{m}{2r}x - \frac{l}{2} = 0$$

17) $$\frac{(l-2z)^2}{m^2} + \frac{y^2}{r^2} = 1.$$

It does not need to explain what these curves represent. The equation of the development is

18) $$z + \frac{m}{2} \sin\frac{\xi}{r} = \frac{l}{2},$$

which is also the well-known curve.

The equation of the heading surface is

5) $$z = g + \frac{m}{\pi} tg^{-1}\frac{y}{x},$$

and those of the heading joint

19) $$\begin{cases} y^2 + x^2 - r^2 = 0 \\ z = g + \dfrac{m}{\pi} tg^{-1}\dfrac{y}{x}. \end{cases}$$

The projections of this curve on the coordinate planes are

20) $$y^2 + x^2 - r^2 = 0,$$

21) $$x = r\cos\frac{\pi(z-g)}{m},$$

22) $$y = r\sin\frac{\pi(z-g)}{m}.$$

These are also familiar curves. The curve 21) touches the line $x=r$ at $z=g$ and $x=-r$ at $z=m-g$, is concave to the axis of x for positive values of x and convex for its negative values, having the point of inflexion at $x=0$. The curve 22) touches the line $y=r$ at $z=\frac{m}{2}-g$, is concave to the axis of z and has no point of inflexion, intersecting the axis of z at $z=g$. The development of the heading joint is

23) $$z + \frac{m}{\pi r}\xi = g + \frac{m}{2},$$

which represents a right line, as it evidently should be.

The equation of the coursing surface, by the aid of 9), is found to be

$$24)\qquad z=h-\frac{\pi r^2}{m}tg^{-1}\frac{y}{x},$$

and those of the coursing joint

$$25)\qquad \begin{cases} y^2+x^2-r^2=0 \\ z=h-\dfrac{\pi r^2}{m}tg^{-1}\dfrac{y}{x}. \end{cases}$$

The projections of this curve on the coordinate planes are

$$26)\qquad y^2+x^2-r^2=0,$$

$$27)\qquad x=r\cos\frac{m(h-z)}{\pi r^2},$$

$$28)\qquad y=r\sin\frac{m(h-z)}{\pi r^2}$$

The curve 27) touches the line $x=r$ at $z=h$ and $x=-r$ at $z=h-\frac{\pi^2 r^2}{m}$ is convex to the axis of x for positive values of x and concave for its negative values, having the point of inflexion at $x=0$. The curve 28) touches the line $y=r$ at the point $z=h-\frac{\pi^2 r^2}{2m}$, is concave to the axis of z and has no point of inflexion, intersecting the axis of z at $z=h$. The development of the coursing joint is

$$29)\qquad z-\frac{\pi r}{m}\xi=h-\frac{\pi^2 r^2}{2m},$$

which is normal to 23), as it should be,

If the extradosal surface be a concentric cylinder to the soffit, and if

$$30)\qquad y^2+x^2-r_1^2=0$$

be its equation we may proceed in the same manner as with the latter. In this case, the condition 11) of stability reduces to, if $\sin\theta=y/r$,

$$31)\qquad y\geqq\frac{4}{\pi}r_1-r.$$

For those arches in which we may put $r=r_1$, the corresponding maximum central angle will be found to be 148° 17′. Again equation 12) gives

$$\cos\chi = \cos\theta\cos\delta,$$

so that

32) $$x = r\cos\chi\sec\delta.$$

II. Modified Helicoidal Arches.

§ 1. *General Considerations.*

If we apply the preceeding investigations to other arches, as catenarian or geostatic, not a little difficulties will be encountered in the determination of the constant a. Of course it may at least be numerically calculated; and perhaps most simply by graphical solutions—in fact this may be sufficient for all our practical purposes. Still the value of a is generally a function of the quantity $g-h$, and this fact offers not a littte inconvenience in practice. All such difficulties will, however, be at once removed by making the coursing surface

33) $$z = h - \frac{\pi}{m}(x^2+y^2)\,tg^{-1}\frac{y}{x},$$

the heading surface remaining the same as before. This surface can be most simply obtained by refering to cylindrical coordinates, in which case

33a) $$z = h - \frac{\pi}{m}\rho^2\theta,$$

which makes the nature of the surface quite evident.

The equations of a coursing joint, its projections on the coordinate planes and its development on a plane parallel to xz can be readily obtained by exactly the same methods as before.

As to the condition of stability, the angle ϕ will be given by the relation

34) $$\cos\phi = \frac{2\theta\cos\theta\cos\delta - \sin\theta\cos\delta + \sigma\sin\delta}{\sqrt{1+\theta^2}},$$

where

$$\sigma = \frac{m}{\pi}(x^2+y^2)^{-\frac{1}{2}},$$

$$\sin\theta = y(x^2+y^2)^{-\frac{1}{2}}.$$

In this case, if $\pi y_0 > \lambda$, y_0 being the value of y for $\theta = \frac{\pi}{2}$, the shears at the crown and the springing will be of opposite senses, but *so long as*

$$|(\cos\phi)_{\theta=\frac{\pi}{2}}| \leqq |(\cos\phi)_{\theta=0}|,$$

which reduces to

$$\lambda \leqq (\pi + \sqrt{4+\pi^2})\, y_0,$$

the arch may be full-centered, and the shears along the coursing beds should nowhere be greater than that at crown.

It is worthy to notice here that the shear at crown in the modified helicoidal arches is *less than* that in the helicoidal arches whenever

$$35) \qquad \pi\sqrt{4+\pi^2}\, y_0\, (am - \lambda y_0) > 2m\, (\pi y_0 - \lambda) \sqrt{a^2 + y_0^2}.$$

Since for circular arches we may put $a = \pi r^2/m$, $y_0 = r, \lambda = 2r$, 35) reduces to

$$r\pi\sqrt{3+\pi^2} > m,$$

and since any skew arch should not be out of this condition, *we conclude that, in this respect of shears, the modified helicoidal circular arches are always more stable than the helicoidal circular.* The condition 12) remains of course the same as before.

§ 2. *Application to a Circular Arch.*

This case can be evidently treated exactly in the same manner as in the foregoing case and no further demonstrations will be necessary, except as to the condition of stability 34), which gives

$$\left| \cos\phi \right|_{\theta=0} = \frac{m}{\pi r} \sin\delta,$$

$$\left| \cos\phi \right|_{\theta=\frac{\pi}{2}} = \frac{m\sin\delta - \pi r\cos\delta}{r\sqrt{4+\pi^2}},$$

and

$$\lambda = 2r < (\pi + \sqrt{4+\pi^2})\, r,$$

so that the arch may be always full-centered *in this respect,* the shear at the springing being of the opposite sense as that at crown, since $\lambda = 2r < \pi r$.

§ 3. *Application to a Transformed Catenarian Arch.*

I shall treat, as an example, the soffit only, which is a transformed catenarian cylinder represented by the equation

$$36) \qquad b - y - a\cosh\frac{x}{n} = 0$$

where

$$n = \frac{\lambda}{\cosh^{-1}\dfrac{b}{a}}$$

The equations of the intrados at the head are

37) $$\begin{cases} b - y - a \cosh \dfrac{x}{n} = 0 \\ h + \dfrac{m}{\lambda} x - \dfrac{l}{2} = 0, \end{cases}$$

and the projections on the coordinate planes are

38) $$b - y = a \cosh \frac{x}{n}$$

39) $$h + \frac{m}{\lambda} x = \frac{l}{2}$$

40) $$b - y = a \cosh \frac{\lambda\left(z - \dfrac{l}{2}\right)}{mn},$$

while the development is

41) $$\begin{cases} \xi = n\left\{ \dfrac{\sqrt{y^2-a^2}\,\sqrt{y^2+n^2-a^2}}{y} + \displaystyle\int_0^{\frac{\sqrt{y^2-a^2}}{y}} \dfrac{dy}{\sqrt{1-y^2 . 1-k^2y^2}} - \displaystyle\int_0^{\frac{\sqrt{y^2-a^2}}{y}} \dfrac{\sqrt{1-k^2y^2}}{\sqrt{1-y^2}}\,dy \right\} \\ z = \dfrac{l}{2} - \dfrac{m}{\lambda} x, \end{cases}$$

in which

$$k = |\sqrt{n^2-a^2}|/n$$

and

$$b - y = a \cosh \frac{x}{n},$$

x and y being parameters,

The equations of the heading joints are

42) $$\begin{cases} b - y = a \cosh \dfrac{x}{n}, \\ z = g + \dfrac{m}{\pi} tg^{-1} \dfrac{y}{x}, \end{cases}$$

and its projections on the coordinate planes

43) $$b - y = a \cosh \frac{x}{n},$$

44) $$z = g + \frac{m}{\pi} tg^{-1} \frac{b - a\cosh\frac{x}{n}}{x}$$

45) $$b - y = a \cosh\left\{\frac{y}{n} tg \frac{\pi(z-g)}{m}\right\}.$$

The curve 44) cuts the line $x = \frac{\lambda}{2}$ at $z = g$, $x=0$ at $z=g+\frac{m}{2}$, $x=-\frac{\lambda}{2}$ at $z=g+m$, is convex to the axis of x for positive values of x and concave for its negative values, the point of inflexion being at the point $(x = 0, z = g+\frac{m}{2})$. The curve 45) may be traced by means of descriptive geometry from the projections on the planes xy and xz. The equations of the development are

46) $$\begin{cases} \xi = n\left\{\dfrac{\sqrt{y^2-a^2}\sqrt{y^2-n^2-a^2}}{y} + \displaystyle\int_0^{\frac{\sqrt{y^2-a^2}}{y}} \frac{dy}{\sqrt{1-y^2 . 1-k^2y^2}} - \int_0^{\frac{\sqrt{y^2-a^2}}{y}} \frac{\sqrt{1-k^2y^2}}{\sqrt{1-y^2}}dy\right\} \\ z = g + \dfrac{m}{\pi} tg^{-1}\dfrac{y}{x}, \end{cases}$$

where x, y, k have the same meanings as before.

The equations of the coursing joints are

47) $$\begin{cases} b - y = a \cosh\dfrac{x}{n} \\ z = h - \dfrac{\pi}{m}(x^2+y^2) tg^{-1}\dfrac{y}{x}, \end{cases}$$

and its projections on the coordinate planes

48) $$b - y = a \cosh\frac{x}{n},$$

49) $$z = h - \frac{\pi}{m}\left[x^2 + \left\{b - a\cosh\frac{x}{n}\right\}^2\right] tg^{-1}\frac{b - \cosh\frac{x}{n}}{x}$$

50) $$z = h - \frac{\pi}{m}\left[y^2 + \left\{n\cosh^{-1}\frac{b-y}{a}\right\}^2\right] tg^{-1}\frac{y}{n\cosh^{-1}\frac{b-y}{a}},$$

while that on the plane yz may be found from other projections by means of descriptive geometry. The development is

$$51)\quad \begin{cases} \xi = n\left\{ \dfrac{\sqrt{y^2-a^2}\,\sqrt{y^2+n^2-a^2}}{y} + \displaystyle\int_0^{\frac{\sqrt{y^2-a^2}}{y}} \dfrac{dy}{\sqrt{1-y^2 \cdot 1-k^2y^2}} - \displaystyle\int_0^{\frac{\sqrt{y^2-a^2}}{y}} \dfrac{\sqrt{1-k^2y^2}}{\sqrt{1-y^2}}\,dy \right\} \\ z = h - \dfrac{\pi}{m}(x^2+y^2)\,tg^{-1}\dfrac{y}{x} \end{cases}$$

x, y, k having the same meanings as above.

No special discussions need be taken up here about the stability of the arch. It may be full-centered if

$$\lambda \leqq (\pi + \sqrt{4+\pi^2})(b-a),$$

which shall be the case with the majority of the arches of this description.

§ 4. *Application to Other Arches.*

The processes hitherto described can be applied to other arches as geostatic, elliptic, etc. exactly in the same manner. The methods being simple enough and the results shewing little peculiarities, no details need here be entered into.

III. Logarithmic Arches.

§ 1. *General Considerations.*

Under this heading are included all those skew arches whose heading surfaces are planes parallel to the heads, and the coursing surfaces are right conoids whose generating line is parallel to the normal cross section of the arch and are guided on one side by the right line in the plane of springing lines equidistant and parallel to them and on the other by the line on the soffit normal to the heading joints, the heads of the arch being supposed to be parallel to each other. With this definition of logarithmic arches, it appears difficult to me to arrive at the analytical solutions of any interest, except in the case of familiar circular arches, of which so much have been already written. In fact the name "logarithmic" cannot be applied to such kinds of arches as catenarian and geostatic, provided the term is to be applied to those kinds of arches defined above. However, as it may not be without interest to inquire what are the general equations of heading and coursing surfaces, I shall try to catch this opportunity to set forth some short descriptions on the subject.

Taking the axes and the notations as in the preceding investigations, the heading surface will be represented by

$$z = g - \frac{m}{\lambda} x \qquad 52)$$

Hence the heading joints, their developments and their projections on the coordinate planes can be discussed exactly in the same manner as with those of the head of the helicoidal arches.

It is a well-known proposition in Geometry that a right conoid can be represented by the equation of the form

$$z = \chi\left(\frac{y}{x}\right).$$

To determine this function from the perpendicularity of its line of intersection with the soffit to the heading joints, it can be shewn without difficulty that, if

$$u = \frac{\phi(x)}{x}$$

$$x = \sigma(u)$$

$$\int_0^x \left|\frac{d\phi}{dx}\right|^2 dx = \tau(u).$$

then

$$z = h + \frac{\lambda}{m}\left\{\sigma\left(\frac{y}{x}\right) + \tau\left(\frac{y}{x}\right)\right\} \qquad 53)$$

is the equation of the coursing surface.

The condition of stability in this general case will not be touched here, as it seems to me not to give an interesting result. The equation 12), however, will of course hold true in this case also.

§ 2. *Application to a Circular Arch.*

No investigations need here be given, except of the coursing surface and the conditions of stability. Here

$$u = \frac{\sqrt{x^2 - x^2}}{x}$$

$$\int_0^x \left|\frac{d\phi}{dx}\right|^2 dx = r \log \frac{r + x}{\sqrt{r^2 - x^2}} - x$$

$$= r\log\frac{1+\sqrt{1+u^2}}{u} - x$$

$$\lambda = 2r.$$

Hence the equation of the coursing surface is

54) $$z = h + \frac{2r^2}{m}\log\frac{x+\sqrt{x^2+y^2}}{y} \quad \text{or} \quad x = y\sinh\frac{m(z-h)}{2r^2}$$

Proceeding in similar manner as in the case of helicoïdal arches, it can be shewn that the equation of the development of the coursing joint is

55) $$z = h + \frac{2r^2}{m}\log tg\left(\frac{\pi}{4}+\frac{\xi}{2r}\right) = h + \frac{2r^2}{m}gd^{-1}\frac{\xi}{2r},$$

and its projections on the coordinate planes

56) $$x^2+y^2=r^2,$$

57) $$z = h + \frac{2r^2}{m}\log\frac{r+\sqrt{r^2-y^2}}{y} \quad \text{or} \quad y = \frac{r}{\cosh\frac{m(z-h)}{2r^2}},$$

58) $$z = h + \frac{r^2}{m}\log\frac{r+x}{r-x} \quad \text{or} \quad x = r\left\{\cosh\frac{m(z-h)}{2r^2} - \sqrt{\coth^2\frac{m(z-h)}{2r^2} - 1}\right\}.$$

The curve 55) cuts the axis of z at $z=h$, is convex to the axis of ξ on its positive side and concave on its negative side, having the lines $\xi=\frac{1}{2}\pi r$ and $\xi=-\frac{1}{2}\pi r$ for its asymptotes. The curve 57) touches the line $y=r$ at $z=h$, is concave to the axis of z, has the point of inflexion at $y=r/\sqrt{2}$, the axis of z being its asymptote. The curve 58) cuts the axis of z at $z=h$, is convex to the axis of x on its positive side and concave on its negative side, having the lines $x=r$ and $x=-r$ for its asymptotes.

As to the condition of stability, we have for the angle ϕ between the coursing surface and the head

59) $$\cos\phi = \frac{\sin\theta - \sigma\cos\theta}{\sqrt{1+\sigma^2 cosec^2\theta}},$$

in which

$$\sigma = \frac{2r^2}{m}(x^2+y^2)^{-\frac{1}{2}}$$

$$\sin\theta = y(x^2+y^2)^{-\frac{1}{2}}$$

Thus *the angle* ϕ *is a right angle for all points on the soffit and on the plane* *of xz. The shears at the crown line is the maximum and continuously decreases up to zero at the springing, so that there is no shear of contrary signs in this kind of arches.* Even this deviation at the crown will be found to be very small, and if we take the maximum value of $\sqrt{x^2+y^2}$ at $\frac{18}{15}r$, and $\delta=30^\circ$, which is certainly an extreme case, we shall have

$$\phi=86^\circ\ 17'.$$

Although the logarithmic arch can thus be fullcentered with almost perfect security in this connection, there is still another condition defined by 12) to limit the arch to segmental. In fact the latter is much more important than in the case of helicoidal or modified helicoidal arches, as the heading surfaces are all of the same kind as the heads.