On the Rivets of a Plate Girder

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The rationality of but a few of the demonstrations on the subject as found in the current text books on bridge buildings has encouraged me to put forth my following investigations on the problem. The discussion is partly in conformity wih the prevailing theories and partly in discrepancy : all have been given here for the sake of convenience.

§ 1. *Fundamental principles.*

All the following deductions are based on this single principle : *The rivets have to sustain the stresses whieh would have existed, had the material been continuous at the joints of two or more pieces for whose connections they are employed.*

This seems to be somewhat unsatisfactory as considered from the theory of elasticity, but no better appears to me to be accessible.

Taking the central axis of the plate girder as the axis of x, the neutral axis of its end cross section as one of y, and the vertical line through the origin, the centre of gravity of the cross section, as that of z, and taking an elementary parallelopiped whose edges are parallel to these rectangular axes, let ξ_x, ξ_y, ξ_z denote the stress intensities on the surface $dydz$ parallel to the axes of x, y, z ; η_x, η_y, η_z, those on $dzdx$; and ζ_x, ζ_y, ζ_z, those on $dxdy$, parallel to the same coordinate axes. Then by the fundamental principles in the common theory of beams we have

$$(1)\quad \xi_x = \frac{Mz}{I}$$

$$(2)\quad \eta_y = 0$$

$$(3)\quad \zeta_z = 0$$

$$(4)\quad \eta_z = \zeta_y = 0$$

$$(5)\quad \zeta_x = \xi_z = \frac{SG_z^h}{yI}$$

$$(6)\quad \xi_y = \eta_x = 0$$

in which

M = bending moment at the section through the point under consideration

S = shear at ditto

I = moment of inertia of the section about the neutral axis

G_z^h = geometrical moment of the section taken between the limits h and z, h being the vertical distance of the remotest element from the axis of y.

y = breadth of the cross section at the point under consideration.

§ 2. *Rivets in Flanges.*

The rivets in flanges have to sustain the shearing stresses arising along the planes of separation of flange plates. Hence the pitch is to be determined both for the plane of separation AB, the lowermost one, and for CD, the smaller value being adopted for the actual pitch. The latter will usually give the smaller pitch, as we shall presently see.

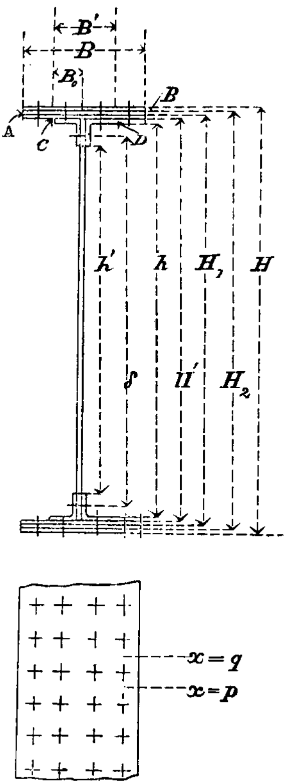

Fig. 1

Let f denote the smaller value of the total shearing and bearing strengths for the assumed rivet diameter and the given flange plate, n the number of rivets in a row at AB (4 in fig.), and m that at CD (2 in fig.). Then assuming all the rivets in a row to be equally stressed, we have

$$(7)\quad nf = \int_q^p \frac{SG_{\frac{H_1}{2}}^{\frac{H}{2}}}{BI}Bdx$$

$$= \frac{G_{\frac{H_1}{2}}^{\frac{H}{2}}}{I}\int_q^p Sdx$$

$$= \frac{B(H^2-H_1^2)}{8I}(M_p-M_q)$$

$$(8)\quad mf = 2\int_q^p \frac{SG_{\frac{H'}{2}}^{\frac{H}{2}}}{B'I}B_1dx$$

$$= \frac{BB_1(H^2-H'^2)}{4IB'}(M_p-M_q),$$

M_p and M_q denoting the value of M at the distances p and q from the origin respectively It will be observed that the above formulae are slightly approximate on the side of danger, as they take no account of the rivet holes existing in the interval of $x=p$ and $x=q$; but the effect is insignificant.

From these formulae it will be evident that to determine the required pitch

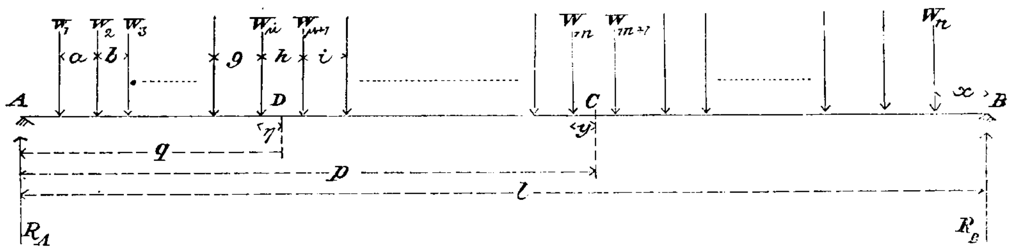

Fig. 2.
The distances PQ and QR are exaggerated,

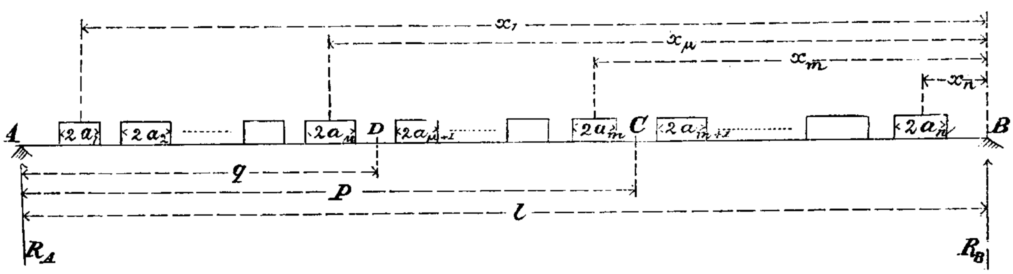

Fig. 3.
The distance CD is exaggerated.

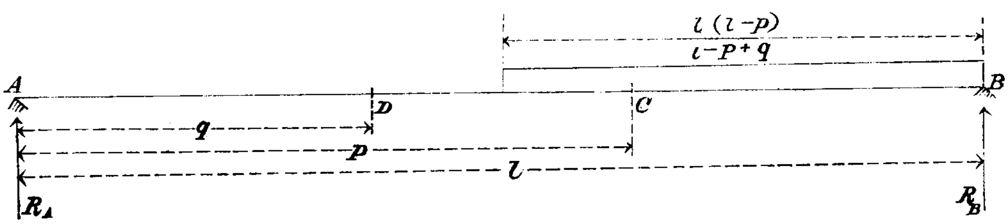

Fig. 4
The distance CD is exaggerated.

we must know the maximum value of $M_p - M_q$. To find it we may proceed as follows : —

1) *Under a concentrated load system.*

Let $W_1, W_2, \cdots\cdots, W_n$ be the concentrated loads as shewn in the figure. Then putting $M_p - M_q \equiv \varDelta$, we have evidently

$$(9)\quad \varDelta = R_A p + [aW_1 + b\sum_1^2 W_r + \cdots + g\sum_1^{\mu-1} W_r + h\sum_1^{\mu} W_r + \cdots + y\sum_1^{m} W_r]$$

$$- R_A q - [aW_1 + b\sum_1^2 W_r + \cdots + g\sum_1^{\mu-1} W_r + \eta\sum_1^{\mu} W_r]$$

$$= \frac{p-q}{l}[aW_1 + b\sum_1^2 W_r + \cdots + x\sum_1^{n} W_r]$$

$$- [(h-\eta)\sum_1^{\mu} W_r + i\sum_1^{\mu+1} W_r + j\sum_1^{\mu+2} W_r + \cdots + y\sum_1^{m} W_r],$$

$\sum_1^n W_r$, for instance, standing for $\sum_{r=1}^{r=n} W_r$.

Hence for maximum of $\varDelta$, we get, since $dx = dy = d\eta$,

$$\frac{d\varDelta}{dx} = \frac{p-q}{l}\sum_1^n W_r + \sum_1^{\mu} W_r - \sum_1^{m} W_r = 0,$$

so that

$$(10)\quad \frac{p-q}{l}\sum_1^n W_r = \sum_{\mu+1}^{m} W_r.$$

This gives the position of the load system for maximum of $\varDelta$, whose value may be determined by (9).

2) *Under a uniform load.*

Let w be the load intensity per unit run. Then

$$(11)\quad \varDelta = 2w\frac{l-p}{l}\sum_1^m a_r(l-x_r) + \frac{2wp}{l}\sum_{m+1}^{n} a_r x_r$$

$$- 2w\frac{l-q}{l}\sum_1^{\mu} a_r(l-x_r) - \frac{2wq}{l}\sum_{\mu+1}^{n} a_r x_r$$

$$= -\frac{2w(p-q)}{l}\sum_1^{\mu} a_r(l-x_r)$$

$$+ \frac{2w}{l}\sum_{\mu+1}^{m} a_r[l(l-p) - (l-p+q)x_r]$$

$$+\frac{2w(p-q)}{l}\sum_{m+1}^{n} a_r x_r.$$

From this relation it appears that Δ is maximum when AD is unloaded, CB is fully loaded, and CD is loaded from C to the point such that

$$l(l-p)-(l-p+q)x_r=0,$$

i. e.,

$$x_r=\frac{(l-p)l}{l-p+q}.$$

Thus we have

$$(12)\quad max\ \Delta=\frac{w}{2}\ \frac{(l-p)^2(p-q)}{l-p+q}.$$

[It may be here observed that some authority says the maximum of Δ to occur, when the shear at C is maximum. But this is faulty, as, for instance, under uniform loading, the latter takes place when the load extends from C to B, in which case

$$(13)\quad \Delta=\frac{w}{2}\ \frac{(l-p)^2(p-q)}{l},$$

which is evidently less than the value in (12). For small values of $p-q$, the error is small, but not for its appreciable values. In such a case only, as at present, in which the value of $p-q$, the pitch of rivets, is small compared with l, the span length, either formula (12) or (13) may be employed without sensible error.]

In conclusion, denoting the required pitch by c, we have :

under a concentrated load system

$$(14)\quad nf=\frac{B(H^2-H_1^2)}{8I}\ \Delta$$

$$(15)\quad mf=\frac{B\,(H^2-H'^2)}{4IB'}\ \Delta,$$

in which

$$\Delta=\frac{p-q}{l}\,[aW_1+b\sum_1^2 W_r+\cdots+x\sum_1^n W_r]$$

$$-[(h-\eta)\sum_1^{\mu} W_r+i\sum_1^{\mu+1} W_r+j\sum_1^{\mu+2} W_r+\cdots+y\sum_1^{m} W_r],$$

under the condition that

$$c\sum_{1}^{n} W_r = l\sum_{\mu+1}^{m} W_r;$$

and under a uniform load

$$(16)\quad nf=\frac{B(H^2-H_1^2)}{16I}\,\frac{(l-p)^2cw}{l-c}$$

$$(17)\quad mf=\frac{BB_0(H^2-H'^2)}{8IB'}\,\frac{(l-p)^2cw}{l-c},$$

whence

$$(18)\quad c=\frac{16nfIl}{B(H^2-H_1^2)(l-p)^2w+16nfI}$$

$$(19)\quad c=\frac{8mfIlB'}{BB_0(H^2-H'^2)(l-p)^2w+8mfIB'}$$

The latter is evidently less than the former in ordinary cases.

§ 3. *Rivets connecting the Angles to Web with no Flange Plate*

There can be no rational formula for the determination of the pitches in this connection Some bridge authority has given a formula depending on the vertical shear, but this is not correct, as the rivets under consideration have to resist the stresses due to η_x, which is by no means affected by the vertical shear however great. The deduction seems to have resulted from the confusion of ζ^x with η_x, the former really depending on the vertical shear, but never affecting the rivets in question.

There is however a fact, apart from the theoretical deductions under the assumptions usually adopted in the common theory of beams, which ought not to be neglected. When the ties or cross girders directly rest on flanges as in railway or highway bridges, there will be some stresses arising from this cause—a consideration not included in the common theory of beams. The quantitative influence of this cause is however very obscure, and some authority recommends to assume the wheel load in a railway bridge to be uniformly distributed for the interval of about three consecutive sleepers. At any rate, some considerations must necessarily be taken into account.

§ 4. *Rivets in Joints of Flanges.*

There is a great obscurity for the determination of rivets required for the joints of flanges. The only rational, but very approximate, means seems to me to be the following : *The rivets required to sustain the stresses arising from the normal stresses (ξ_x) on the joints shall be driven, at any allowable pitches, in addition to the proper rivets for flanges, the effect of tangential stresses (ξ_z) being neglected.*

For instance let the three flange plates be jointed at the sections P, Q and R. Denoting the *simultaneons* values of moments at the sections throngh P, Q

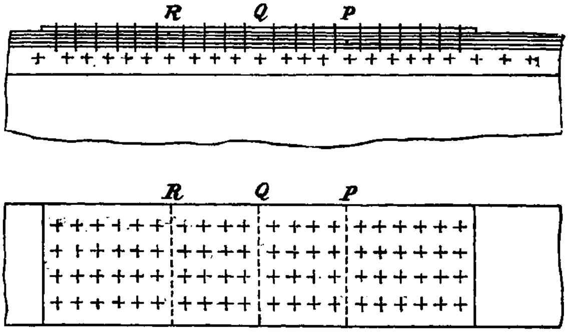

Fig. 5.

and R by M_1, M_2 and M_3, we have to drive the rivets, at any allowable pitches on each side of the joints P, Q and R, determined by,

$$\frac{1}{f}\int_{\frac{H'}{2}}^{\frac{H_1}{2}} \frac{M_1 z}{I} B dz = \frac{BM_1}{8fI}(H_1^2 - H'^2)$$

$$\frac{1}{f}\int_{\frac{H_1}{2}}^{\frac{H_2}{2}} \frac{M_2 z}{I} B dz = \frac{BM_2}{8fI}(H_2^2 - H_1^2)$$

$$\frac{1}{f}\int_{\frac{H_2}{2}}^{\frac{H}{2}} \frac{M_3 z}{I} B dz = \frac{BM_3}{8fI}(H^2 - H_2^2).$$

The total number of rivets on each side of the joints is then

$$(20)\quad \frac{B}{8fI}[(H_1^2 - H'^2)M_1 + (H_2^2 - H_1^2)M_2 + (H^2 - H_2^2)M_3].$$

That this may be maximum we must have, as can be easily proved,

$$(21)\quad \frac{(H_1^2 - H'^2)p + (H_2^2 - H_1^2)q + (H^2 - H_2^2)r}{l}\sum_1^m W_r$$

$$= (H_1^2 - H'^2)\sum_1^m W_r + (H_2^2 - H_1^2)\sum_1^\nu W_r + (H^2 - H_2^2)\sum_1^\mu W_r.$$

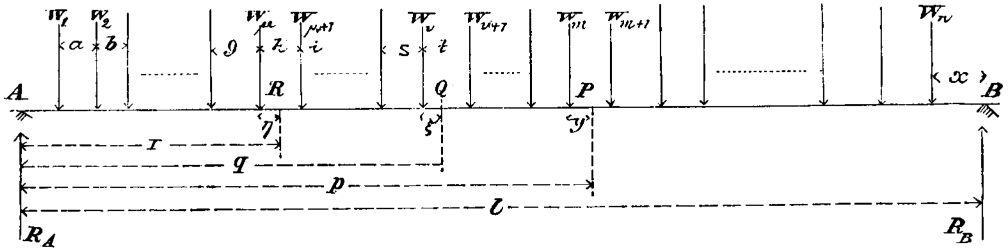

Fig. 6

The distance between C and D is exaggerated.

There may be of course several positions of the load system, each of whieh giving a maximum, the greatest of which will be the required maximum.

Under a uniform load, M_1, M_2, and M_3 take their maximum values simultaneously under the whole loading, so that (20) assumes the value

$$(21)\ \frac{wB}{16fI}[(H_1^2-H'^2)p(l-p)+(H_2^2-H_1^2)q(l-q)+(H^2-H_2^2)r(l-r)].$$

§ 5. *Rivets in Joints of Web.*

Assuming, as is usually done, that the stress on a rivet used in this connection is proportional to its distance from the neutral axis, we have, neglecting the

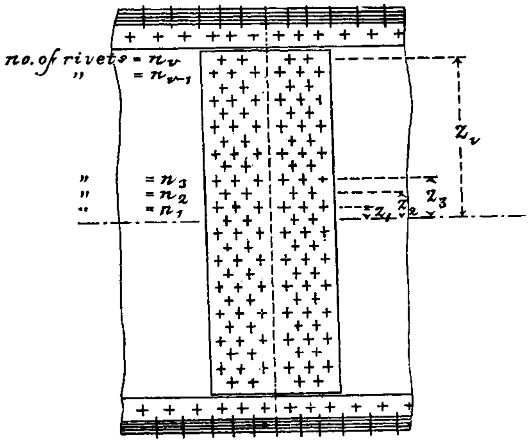

Fig. 7.

effects of rivets connecting the web to the angles,

$$(22)\ f_r = \frac{1}{2}\sqrt{\frac{M_0^2 z_r^2}{[\sum_1^v n_r z_r^2]^2} + \frac{S_0^2}{[\sum_1^v n_r]^2}},$$

where

f_r = total stress on a rivet at the distance z_r from the neutral axis

M_0 = bending moment sustained by the *web only*

S_0 = *simultaneous* shear sustained by the *web only*.

The stress is consequently maximum on the rivet at the remotest distance z_v from the neutral axis, its value being

* This seems to be at least doubtful.

$$(23)\ f_v = \tfrac{1}{2}\sqrt{\frac{M_0^2 z_v^2}{[\sum_1^v n_r z_r^2]^2} + \frac{S_0^2}{[\sum_1^v n_r]^2}}.$$

[It may be here remarked, in connection with this formula that, whatever assumptions were taken for the design of flanges and web, the stress distributions, as determined by the theory of beams, will by no means be altered, so that even in such a case, as is sometimes done, that the flange areas are determined by the moment and the web area by the shear alone, the effects of both the normal and tangential stresses ought to be strictly taken into consideration for the determination of rivets in question.]

In practice, we have commonly

$$n_1 = n_3 = n_5 = \cdots\cdots$$

and

$$n_2 = n_4 = n_6 = \cdots\cdots$$

Hence *when* v *is even and* $n_1 = n_2 \pm 1$,

$$(24)\ \begin{cases} \sum_1^v n_r z_r^2 = \dfrac{v(v+1)z_1^2}{6}[n_1(2v+1) \mp (v+2)] \\ \sum_1^v n_r \quad = \dfrac{v}{2}(2n_1 \mp 1); \end{cases}$$

when v *is odd and* $n_1 = n_2 \pm 1$,

$$(25)\ \begin{cases} \sum_1^v n_r z_r^2 = \dfrac{v(v+1)z_1^2}{6}[n_1(2v+1) \mp (v-1)] \\ \sum_1^v n_r \quad = \dfrac{v}{2}(2n_1 \mp 1) \pm \tfrac{1}{2}; \end{cases}$$

and *when* $n_1 = n_2$, v *being even or odd,*

$$(26)\ \begin{cases} \sum_1^v n_r z_r^2 = \dfrac{v(v+1)z_1^2}{6} n_1(2v+1) \\ \sum_1^v n_r \quad = n_1 v. \end{cases}$$

Theoretically, M_0 and S_0 being each a function of the distance (say x) of the last load to the right hand point of support, (23) is an equation of the form

$$\phi(x \mid v) = 0,$$

so that the maximum of v may be determined by the equations

$$\frac{\partial \phi}{\partial x} = 0 \text{ and } \phi' x \mid v) = 0.$$

But as the form of the function ϕ is so complicated as not to admit a simple solution of these equations,* it can be solved but very approximately. For this purpose, find the maximum value of M_0 and the *simultaneous* value of S_0, and also the maximum value of S_0 and the *simultaneous* value of M_0. Find, by trial, the requisite value of v in each case, and adopt the greater value. As we cannot foresee at outset which of (24)—(26) is to be employed, some judgement and guess will be required for the choice.

§ 6. *Rivets in Joints of Angles.*

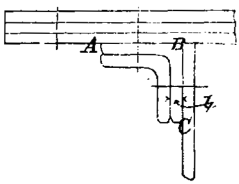

Fig. 8.
For other symbols refer to fig. 1.

The calculation of the rivets in the joint of angle offers the most difficult task, and no satisfactory method, even approximately, has yet been discovered. For *provisional* method, the tangential stresses beingneglected, we may proceed as follows : the more rigorous treatment seems to require a further development of the theory of elasticity.

Let n denote the number of rivets in the leg AB and m that in the leg BC. Then

$$nf\frac{h_0}{2} + mf\frac{\delta}{2} = t\int_{\frac{h'}{2}}^{\frac{h}{2}} \frac{Mz}{I} z\,dz + B_0\int_{\frac{h}{2}}^{\frac{H'}{2}} \frac{Mz}{I} z\,dz$$

$$= \frac{M}{24I}\left[t(h^3-h'^3)+B_0(H'^3-h^3)\right]$$

$$nf+mf = t\int_{\frac{h'}{2}}^{\frac{h}{2}} \frac{Mz}{I}dz + B_0\int_{\frac{h}{2}}^{\frac{H'}{2}} \frac{Mz}{I}dz$$

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* It will be seen that (23) may be put in the form

$$\frac{(m+nx-py)^2}{[v(v+1)(\alpha v+\beta)]^2} + \frac{(q+rx)^2}{(\gamma v+\delta)^2} = 1,$$

where $m,n,p,q,r,\alpha,\beta,\gamma$ and δ are all independent of x,y and v; or putting

$$[v(v+1)(\alpha v+\beta)]^2 \equiv \psi(v)$$

and

$$(\gamma v+\delta)^2 \equiv \chi(v),$$

we have

$$\frac{(m+nx-py)^2}{\psi(v)} + \frac{(q+rx)^2}{\chi(v)} = 1.$$

Hence the equations $d\phi/dx$ and $\phi\ (x\,|\,v)=0$, give

$$\begin{cases} \dfrac{(n-p)(m+nx-py)}{\psi(v)} + \dfrac{r(q+rx)}{\chi(v)} = 0 \\ \dfrac{(m+nx-py)^2}{\psi(v)} + \dfrac{(q+rx)^2}{\chi(v)} - 1 = 0, \end{cases}$$

the solution of which would require one of equations of twelfth degree in v.

$$= \frac{M}{8I}[t(h^2 - h'^2) + B_0(H'^2 - h^2)],$$

in which M is the total bending moment at the section, h_0 being equal to H' when f is the shearing strength, and to $H' - t$ when it is the bearing strength for the assumed rivet diameter and the given plate thickness.

Hence putting

$$\frac{M}{12fI}[t(h^3 - h'^3) + B_0(H'^3 - h^3)] \equiv A$$

$$\frac{M}{8fI}[t(h^2 - h'^2) + B_0(H'^2 - h^2)] \equiv B,$$

we have

$$(27)\quad n = \frac{A - \delta B}{h_0 - \delta}$$

$$(28)\quad m = \frac{h_0 B - A}{h_0 - \delta}.$$

Of these values the greater one is to be chosen, and used, at any allowable pitches, in addition to the rivets otherwise properly needed.

§ 7. *Rivets connecting the Angles to Web with Flange Plates.*

In this case, the pitch of the rivets in question shall be determined so as to sustain the shear between the flange plates and the web. Thus, adopting the notations in § 2, we have

$$2f = \int_q^p \frac{S G_{\frac{H'}{2}}^{\frac{H}{2}}}{B'I} \tau\, dx = \frac{B\tau(H^2 - H'^2)}{8IB'} \Delta$$

$$f = \frac{B\tau(H^2 - H'^2)}{8IB'} \Delta,$$

according as f is the shearing or bearing strength for the assumed rivet diameter and plate thickness, τ being the thickness of the web The pitch thus found will however be too great for practical purposes.