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APPLICATION OF THE MARKOV CHAIN ON PROBABILITY OF EARTHQUAKE OCCURRENCE

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1. PREFACE

Probabilistic model of earthquake occurrence is very important to estimate the duration of structural life and the expected future loading on structures in civil engineering field. The most general and widely used model is either the Poisson process or the Markov chain. While the Poisson process is fundamental of all stocastic processes, it is too simple to represent various physical aspects of the phenomenon. On the other hand, the Markov chain is advantageous to include complicated physical aspects in it. The application of the Markov chain on earthquake occurrence is feasible to represent the persistency, nonhomogeneity and periodicity of earthquake occurrence, if any. If earthquake occurrence is not completely incidental and random, but depends on the past earthquake occurrence and the causality in natural phenomenon, application of the Markov chain model is most profitable to analyze them. Though many research works on probabilistic model of earthquake occurrence have been carried out by using the Poisson model in the past[6)~8)], there are very few works on the application of the Markov chain. It will be due to the reason that the persistency, nonhomogeneity and periodicity of earthquake occurrence have not been completely accertained yet.[9)]

It is the purpose of this paper to describe the various applications of the Markov process to estimate the relation between duration of structural life and expected seismic loading on structures. Though the earthquake occurrence is regarded as homogenious and stationary in this paper, which will be reduced to the Poisson process in itself, the application of the Markov chain model will be most effective, when these factors can be taken into consideration in the estimation of earthquake occurrence.

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2. MODEL FOR RANDOM EARTHQUAKE OCCURRENCE

Random occurrence of an event in time is no more than the problem that any numbers of point will lie at random in time axis. In order to apply the Markov process to analyze it, it is necessary to make a model on the number of occurrences of an event in a time interval $(0, t)$.

When the number is denoted by $X(t)$, then $X(t)$ is a discontinuous stochastic process taking integral values,

$$0, 1, 2, \cdots\cdots k, \cdots\cdots .$$

Assuming that present number $X(t_0)$ is known and that the number of occurrences in an interval (t_0, t) is independent of the past value $X(t)$, then $X(t)$ will be expressed by the Markov chain. If the element of one-step transition matrix of the Markov chain $\Pi t\Delta t)$ is denoted by $P_{ij}(\Delta t)$, $P_{ij}(\Delta t)$ means the probability of transition from the number i to the number j in the time interval Δt.

$$P_{ij}(\Delta t)=P\{X(t+\Delta t)=j|X(t)=i\} \cdots\cdots\cdots\cdots (1)$$

where Δt is determined by dividing the time interval $(0, t)$ into N. Since $X(t)$ can only increase, therefore,

$$P_{ij}(\Delta t)=0 \qquad (i>j) \cdots\cdots\cdots\cdots (2)$$

Assuming that one event occurs at most in this time interval Δt, each element of transition matrix can be determined completely by

$$\left.\begin{array}{l} P_{ij}(\Delta t)=1-p \\ P_{i,i+1}(\Delta t)=p \\ P_{NN}(\Delta t)=1 . \end{array}\right\} \quad (i=0, 1, \cdots N-1) \cdots (3)$$

where p is the probability of occurrence in the time interval Δt. It is obvious that all other P_{ij}'s are zeros except these elements. Each element of the transition matrix after n-step, $\Pi(n\Delta t)$ can be easily calculated by multiplying one-step transition matrix $\Pi(\Delta t)$ n times.

$$
\left.\begin{aligned}
&P_{ij}(n\Delta t)=\binom{n}{j-i}(1-p)^{n-j+i}p^{j-i} \\
&\qquad\qquad \begin{pmatrix} n>j-i>0 \\ i=0,1,\cdots N-1 \\ j=0,1\cdots N-1 \end{pmatrix} \\
&P_{ij}(n\Delta t)=0 \qquad \begin{pmatrix} j-i>n,\ j-i<0 \\ i=0,1,\cdots N-1 \\ j=0,1,\cdots N-1 \end{pmatrix} \\
&\text{In } j=N, \\
&P_{iN}(n\Delta t)=p^{N-i}\sum_{r=0}^{n-N+i}\binom{r+N-i-1}{r} \\
&\qquad\qquad \cdot(1-p)^r \\
&\qquad\qquad \begin{pmatrix} i\geq N-n \\ i=0,1,\cdots N-1 \end{pmatrix} \\
&P_{iN}(n\Delta t)=0 \qquad \begin{pmatrix} i<N-n \\ i=0,1,\cdots N-1 \end{pmatrix} \\
&P_{NN}(n\Delta t)=1\,.
\end{aligned}\right\}\cdots(4)
$$

It is clear that the n-step transition matrix is triangular matrix.

Given the unconditional probability of first step by the following row vector

$$P(0)=(p_0(0), p_1(0), \cdots p_N(0))\,, \quad \cdots\cdots\cdots\cdots(5)$$

the state vector after n-step will be determined by the relation

$$
\begin{aligned}
P(n\Delta t)&=P(0)\Pi(n\Delta t) \\
&=(p_0(n\Delta t), p_1(n\Delta t), \cdots p_N(n\Delta t)\,, \cdots(6)
\end{aligned}
$$

where

$$p_j(n\Delta t)=\sum_{i=0}^{N}p_i(0)P_{ij}(n\Delta t)\,. \quad \cdots\cdots\cdots\cdots\cdots\cdots(7)$$

Eq. (7) shows the probability of the number of occurrence after $t=n\Delta t$ when the initial unconditional probability is given. When the number of occurrence in initial state is assumed equal to 0, and

$$P\{X(0)=0\}=1\,, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(8)$$

then, the initial state vector of unconditional probability will be

$$P(0)=(1, 0, 0, \cdots\cdots 0)\,. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(9)$$

It is easy to calculate the probability of number of occurrences after n-step by substituting Eq. (9) into Eq. (7).

$$
\begin{aligned}
P(n\Delta t)&=P\{X(n\Delta t)=j|X(0)=0\} \\
&=\left\{(1-p)^n, \binom{n}{1}(1-p)^{n-1}p,\right. \\
&\quad\left.\cdots\binom{n}{n-1}(1-p)p^{n-1}, p^n, 0, \cdots\right\}\,. \\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(10)
\end{aligned}
$$

Eq. (10) shows the probability of taking the number j ($j=0, 1, \cdots N$) after $t=n\Delta t$ under the condition of no occurrence in $t=0$. The second line of Eq. (10) is nothing but the binomial distribution. It will approach the Poisson distribution when np is of the order 1 and $p\ll 1$.

When this model is applied to the probability of earthquake occurrences above a certain level of intensity, it is necessary to combine it with the distribution of intensity scale level. If arrival rate of whole earthquakes λ_0 is known, the arrival rate of earthquakes which exceed a certain level of intensity I, λ_I, may be obtained with the probability mass function on intensity scale.

$$\lambda_I=\lambda_0\sum_{i=I}^{s}C_i \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(11)$$

in which

- λ_I: Arrival rate of the earthquakes which exceed intensity I
- λ_0: Arrival rate of whole earthquakes.
- C_I: Probability mass function on intensity $I\left(\sum_{i=1}^{s}C_i=1\right)$
- s: Total number of levels

Once the arrival rate of each level, λ_I, is determined, the probability of earthquake occurrences in time interval Δt which exceed the level I is given by the equation,

$$p=\lambda_I\Delta t \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(12)$$

Substituting Eq. (12) into Eq. (4), all elements of transition matrix in time interval $(0, n\Delta t)$, $\Pi(n\Delta t)$, will be determined. Therefore, the probability of exceedence above level I in earthquake occurrence is expressed by using Eq. (10) as follows.

$$
\begin{aligned}
P\{X_I(n\Delta t)\geq 1|X(0)=0\}&=1-P\{X_I(n\Delta t) \\
&=0|X(0)=0\} \\
&=1-(1-\lambda_I\Delta t)^n \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(13)
\end{aligned}
$$

where $X_I(n\Delta t)$ is the number of occurrence which exceeds the level I during $t=n\Delta t$.

According to the seismic reports published by the Japanese Meteorological Agency[3], 297 earthquakes above intensity scale (JMA) III occurred

Table 1 Numbers of Earthquakes in Tokyo from 1898 to 1970.

Intensity Scale (JMA)	III	IV	V	VI	Total
number	249	37	10	1	297
%	83.8	12.5	3.4	0.3	100

$\lambda_0=0.3436$/month

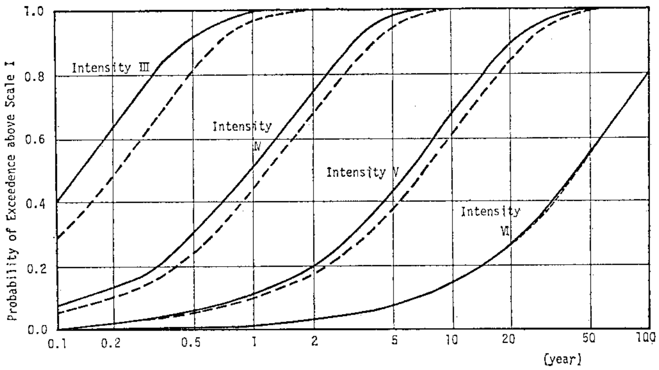

Fig. 1 Probability of Exceedence above Scale I.

in the past in Tokyo district during 72 years from 1898 to 1970. These are classified in **Table 1.** In the table the mean arrival rate of earthquakes is 0.3436/month during these 72 years. The most fitted lines between the JMA Intensity Scale and the percentage of the earthquakes is given by the relation;

$$\ln Y = -1.786X + 9.781 \quad \cdots\cdots(14)$$

where

X: JMA Intensity Scale

Y: Percentage of earthquake occurrence

From Eq. (14) the probability mass function and average arrival rates of earthquake will be calculated as shown in **Table 2.**

Table 2 Probability Mass Function and Average Arrival Rate.

Intensity Scale (JMA)	III	IV	V	VI	Total
C_i	0.833	0.140	0.023	0.004	1.00
$\lambda_i=\lambda_0 C_i$	0.2863	0.0480	0.0080	0.0013	0.3436
$\lambda_I=\sum_{i=1}^{n}\lambda_i$	0.3436	0.0573	0.0093	0.0013	

By taking $\Delta t=1$ month and substituting these data into Eq. (13) the probability which exceeds a certain intensity scale was calculated and plotted in **Fig. 1.** It shows that in Tokyo the earthquakes above Scale III, IV and V will be probable to occur every 1 year, 10 years and 50 years, respectively. The broken lines indicates the same probability calculated by the Poisson model.

$$P\{X_I(n\Delta t)\geq 1\}=1-e^{-\lambda_I n\Delta t} \quad \cdots\cdots(15)$$

The difference between solid lines and broken lines depends on the error between discrete model of the Markov chain and continuous model of the Poisson process as discussed in the Reference[2].

3. MODEL FOR PROBABILITY OF EXCEEDENCE ABOVE A CERTAIN LEVEL

When the events consist of several classified levels, the probability of occurrence of these levels can be easily formulated by the following Markov chain model.

Let $q_{ij}(\Delta t)$ be the transition probability from level i to j in time interval Δt. One-step transition matrix $\Pi(\Delta t)$ is represented by

$$\Pi(\Delta t)=[q_{ij}(\Delta t)] = \begin{bmatrix} q_{00}, & q_{01}, & \cdots\cdots q_{0,s-1} \\ q_{10}, & q_{11}, & \cdots\cdots q_{1,s-1} \\ \cdots & \cdots & \cdots \\ q_{s-1,0}, & q_{s-1,1}, & \cdots\cdots q_{s-1,s-1} \end{bmatrix}, \quad \cdots\cdots(16)$$

where s indicates the total number of levels. After $t=n\Delta t$, n-step transition matrix takes the form,

$$\Pi(n\Delta t)=\Pi(\Delta t)^n=[q_{ij}(n\Delta t)] = \sum_{a=0}^{s-1}\sum_{b=0}^{s-1}\cdots\cdots\sum_{u=0}^{s-1}\sum_{v=0}^{s-1} q_{ia}q_{ab}\cdots\cdots q_{uv}q_{vj}. \quad \cdots\cdots(17)$$

Though the element $q_{ij}(n\Delta t)$ in Eq. (17) means the transition probability from the level i at $t=0$ to the level j at $t=n\Delta t$, it is not clear how it reaches from the level i to the level j in time interval $(0, n\Delta t)$. It will be explained by the Shanon transition chart shown in **Fig. 2.**

As an example, if initial level is assumed level 0 at $t=0$ and final level after $t=n\Delta t$, level 2, in **Fig. 2,** various enroutes can be found to arrive at Level 2 from Level 0 in time interval $(0, n\Delta t)$. There is no saying that the transition from Level 0 to Level 2 has not experienced Level 1 or Level 3 on the way to Level 2. Therefore, it should be

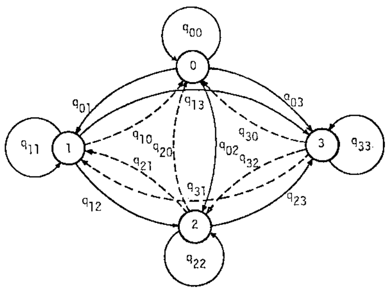

Fig. 2 Shanon Transition Chart ($s=4$).

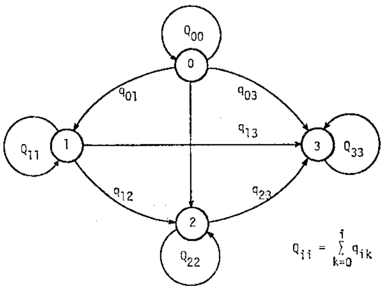

Fig. 3 Shanon Transition Chart for Calculation of Probability below a Certain Level.

noted that $q_{ij}(n\Delta t)$ is no more than the probability starting from initial state i to final state j without regard to its enroute.

In order to apply it on the calculation of the probability of earthquake occurrence, it is necessary to take into account of its past experience until the final level at $t=n\Delta t$. In the example mentioned above, transition should not contain Level 3 on the way to Level 2 to calculate the probability of earthquake occurrences below level 2. If it would have the experience of Level 3 on the way to Level 2, it might not be the probability below Level 2 but the probability below Level 3. On the contrary, even if it experiences Level 2, Level 1 or Level 0 on the way to Level 2, it has no influences on the calculation of probability below Level 2. These situations may be illustrated by the Shanon chart in **Fig. 3**. From this chart the one-step transition matrix to calculate the probability of occurrence below a certain level can be shown by the following upper triangular matrix.

$$\Pi(\Delta t)=\begin{bmatrix} Q_{00}, & q_{01}, & q_{02}, & q_{03}, & \cdots\cdots & q_{0,s-1} \\ 0, & Q_{11}, & q_{12}, & q_{13}, & \cdots\cdots & q_{1,s-1} \\ 0, & 0, & Q_{22}, & q_{23}, & \cdots\cdots & q_{2,s-1} \\ 0, & \cdots\cdots & & Q_{ii}, & q_{i,i+1}, \cdots & q_{i,s-1} \\ 0, & \cdots\cdots & & & & 1 \end{bmatrix} \quad \cdots\cdots(18)$$

where

$$Q_{ii}=\sum_{k=0}^{i} q_{ik} \quad \cdots\cdots(19)$$

Off-diagonal elements above $i<j$ in Eq. (18) indicate the probability of transition to higher level, and diagonal elements indicate the probability staying at the same level. It is obvious from **Fig. 3** that whenever initial state will move lower level at each step, it should be regarded as staying in the initial state. Introducing Eq. (18) into Eq. (17), the n-step transition matrix for the probability of earthquake occurrences, $\Pi(n\Delta t)$, can be derived.

When Eq. (16) indicates limiting probability in ergodic case and q_{ij} and Q_{ii} can be simply represented by q_j and Q_i, respectively, the n-step transition matrix is easily formulated by the equation,

$$\Pi(n\Delta t) = \begin{bmatrix} Q_0{}^n, & q_1 \sum_{m=1}^{n} Q_0^{m-1} Q_1^{n-m}, & \cdots & q_{s-1} \sum_{m=1}^{n} Q_{s-2}^{m-1} Q_{s-1}^{n-m} \\ 0, & Q_1^n, & \cdots\cdots & \vdots \\ 0, & \cdots\cdots Q_i^n, & q_i \sum_{m=1}^{n} Q_{i-1}^{m-1} Q_i^{n-m}, \cdots\cdots & \vdots \\ 0, & \cdots\cdots & & 1 \end{bmatrix} \quad \cdots\cdots(20)$$

Each element in Eq. (20) is transformed into simpler forms by the following relation.

$$\begin{aligned} q_j \sum_{m=1}^{n} Q_{j-1}^{m-1} Q_j^{n-m} &= \sum_{m=1}^{n} Q_{j-1}^{m-1} Q_j^{n-m} (Q_j - Q_{j-1}) \\ &= Q_j^n \sum_{m=1}^{n} (Q_{j-1}^{m-1} Q_j^{-(m-1)} - Q_{j-1}^m Q_j^{-m}) \\ &= Q_j^n \left\{ \sum_{m=0}^{n-1} \left(\frac{Q_{j-1}}{Q_j} \right)^m - \sum_{m=1}^{n} \left(\frac{Q_{j-1}}{Q_j} \right)^m \right\} \\ &= Q_j^n \left\{ 1 - \left(\frac{Q_{j-1}}{Q_j} \right)^n \right\} = Q_j^n - Q_{j-1}^n . \end{aligned} \quad \cdots\cdots(21)$$

Hence, Eq. (20) can be shown simply by

$$\Pi(n\Delta t) = \begin{bmatrix} Q_0^n, & (Q_1^n - Q_0^n), & (Q_2^n - Q_1^n), & \cdots\cdots & (1 - Q_{s-2}^n) \\ 0, & Q_1^n, & (Q_2^n - Q_1^n), & \cdots\cdots & (1 - Q_{s-2}^n) \\ 0, & \cdots\cdots & Q_2^n, & \cdots\cdots & (1 - Q_{s-2}^n) \\ 0, & \cdots\cdots & & & 1 \end{bmatrix} . \quad \cdots\cdots(22)$$

If unconditional probabilities at $t=0$ and at $t=n\Delta t$ are denoted by the row vectors, $P(0)$ and $P(n\Delta t)$, respectively, relation between $P(0)$ and $P(n\Delta t)$ is given by the equation,

$$P(n\Delta t) = P(0)\Pi(n\Delta t) . \quad \cdots\cdots(23)$$

To calculate the transition probability from the level i after $t=n\Delta t$, $P(0)$ will be given by

$$P(0)=(\underbrace{0,\ 0,\ \cdots 1}_{i\text{ th}},\ 0,\ \cdots 0)\ . \qquad\cdots\cdots(24)$$

Substitution of Eq. (24) into Eq. (23) shows that $P(n\Delta t)$ is indicated by the ith row of $\Pi(n\Delta t)$. When q_{ij} and Q_{ii} is replaced by q_j and Q_i in ergodic case, $P(n\Delta t)$ is given simply by

$$P\{I(n\Delta t)=j|I(0)=i\}=Q_j^n-Q_{j-1}^n\ , \quad (j>i) \qquad\cdots\cdots(25)$$

$$P\{I(n\Delta t)\leq i|I(0)=i\}=Q_j^n\ . \qquad\cdots\cdots(26)$$

Eq. (25) is the transition probability from lower intensity level to upper intensity level, and Eq. (26) is the probability staying at the same intensity level as that of present time.

The probability of exceedence above level I $(I>i)$ will take the form.

$$\begin{aligned}P\{I(n\Delta t)>I|I(0)=i\}&=\sum_{j=I}^{s-1}(Q_j^n-Q_{j-1}^n)\\&=Q_{s-1}^n-Q_{I-1}^n\\&=1-Q_{I-1}^n\\&=1-\left(\sum_{j=0}^{I-1}q_j\right)^n. \quad (I>i)\end{aligned} \qquad\cdots\cdots(27)$$

In case of earthquake occurrence q_j is related to the arrival rate λ_0 by the following relation.

$$q_j=\lambda_0 C_j\Delta t \qquad (j=1,\ 2,\ \cdots s-1) \qquad\cdots\cdots(28)$$
$$q_0=1-\lambda_0\Delta t$$

Then, by use of Eq. (11),

$$\begin{aligned}\sum_{j=0}^{I-1}q_j&=1-\sum_{j=I}^{s-1}q_j=1-\lambda_0\sum_{j=I}^{s-1}C_j\Delta t \qquad\cdots\cdots(29)\\&=1-\lambda_I\Delta t\end{aligned}$$

Substituting it into Eq. (27),

$$P\{I(n\Delta t)\geq I|I(0)=i\}=1-(1-\lambda_I\Delta t)^n, \quad (I>i) \qquad\cdots\cdots(30)$$

When $i=0$ in initial state, it is obviously the same equation as Eq. (13). It is noteworthy that in this case the level i has no influences on the result of calculation.

Fig. 4 shows the numerical calculation of Eqs. (25) and (26) by using the data shown in **Table 2.** The solid lines in the figure show the transition probability from lower intensity level to the higher intensity level which is given by Eq. (25). It means the probability that the earthquake having intensity I will occur at least one during x years under the condition of no earthquake occurrence above I+1. For example, the probability that the earthquake of intensity V will occur at least one under the condition of no occurrence above intensity VI will become its maximum after 20 years from now. Since lower level will move to the higher level as time passes, the transition probability of the intensity level V is gradually shifted to that of intensity VI. It suggests that the proper design load for earthquake should be selected to build a structure in accordance with its structural life. In this case the transition probability of intensity V is the highest of all from 7 years till 44 years and after that it is exceeded by that of intensity VI. If one wants to build a structure having structural life from 7 years to 44 years in Tokyo area, for instance, it will be economical to design it against the earthquake of intensity V. These period in each intensity level is listed in **Table 3.** It should be kept in mind that this probability

Table 3 Economical Period of Structural Life Compared with the Intensity Level of Earthquake.

Intensity Scale (JMA)	III	IV	V	VI
Economical Period	— 1.3 years	1.3 years — 7.0 years	7.0 years — 44 years	44 years —

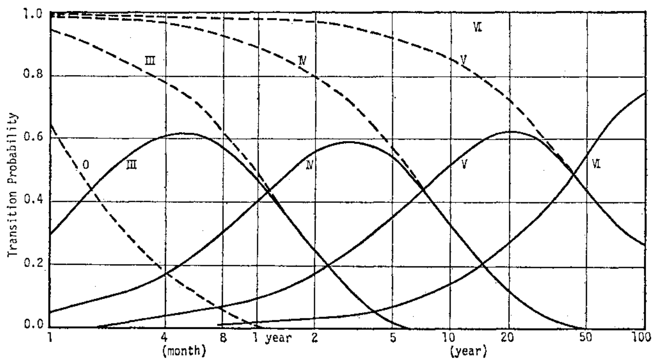

Fig. 4 Transition Probability of Earthquake Occurrence.

does not mean the absolute safety of structures.

Dotted lines in **Fig. 4** show the probability of staying at the same intensity level as that of present time, which is given by Eq. (26).

4. MODEL ON FIRST PASSAGE TIME OF A CERTAIN LEVEL

It is significant to know the reoccurrence time and reoccurrence probability related to the magnitude of earthquakes. It goes without saying that the Markov process is the most suitable model for expressing these relations. In this section the fundamental concepts will be introduced firstly and secondly applied to estimate the earthquake reoccurrence time and probabilities in a most suitable forms.

The conditional probability of being in the level j at time $t=n\Delta t$ under the condition that it was in the level i at $t=0$ and has not been in j before $t=n\Delta t$ will be defined as the first passage time probability in the Markov process.

$$\left.\begin{array}{l} P\{I(k\Delta t)\neq j,\ k=1,2,\cdots n-1, \\ I(n\Delta t)=j\,|\,I(0)=i\}=f_{ij}^{(n)}(s)\,. \end{array}\right\} \quad \cdots\cdots\cdots(31)$$

in which s of $f_{ij}^{(n)}(s)$ indicates the size of the $s\times s$ transition matrix in Eq. (16). The relation between this conditional probability $f_{ij}^{(n)}(s)$ and the original transition probability q_{ij} given in Eq. (16) will be obtained easily by use of Eq. (17).

$$\begin{aligned} f_{ij}^{(n)}(s) &= \sum_{a\neq j}^{s-1}\sum_{b\neq j}^{s-1}\cdots\cdots\sum_{u\neq j}^{s-1}\sum_{v\neq j}^{s-1} q_{ia}q_{ab}\cdots\cdots q_{uv}q_{vj} \\ &= \sum_{a=0}^{s-1}\sum_{b\neq j}^{s-1}\cdots\cdots\sum_{u\neq j}^{s-1}\sum_{v\neq j}^{s-1} q_{ia}q_{ab}\cdots\cdots q_{uv}q_{vj} \\ &\quad -q_{ij}\sum_{b\neq j}^{s-1}\cdots\cdots\sum_{u\neq j}^{s-1}\sum_{v\neq j}^{s-1} q_{jb}\cdots\cdots q_{uv}q_{vj} \\ &= \sum_{a=0}^{s-1} q_{ia}f_{aj}^{(n-1)}(s)-q_{ij}f_{jj}^{(n-1)}(s)\,. \quad \cdots(32) \end{aligned}$$

The regressive difference equation of Eq. (32) is reduced to the following simpler form.

$$f_{ij}^{(n)}(s)=q_{ij}^{(n)}-\sum_{m=1}^{n-1} q_{ij}^{(m)}f_{ij}^{(n-m)}(s) \quad \cdots\cdots\cdots\cdots(33)$$

where $f_{ij}^{(1)}(s)=q_{ij}$ and $q_{ij}^{(n)}=q_{ij}(n\Delta t)$.

When the final state after $t=n\Delta t$ is in the level i, $f_{ii}^{(n)}(s)$ indicates the probability to return level i for the first time after $t=n\Delta t$, which is defined as the first return time probability.

$$\begin{aligned} f_{ij}^{(n)}(s) &= P\{I(k\Delta t)\neq i,\ k=1,2,\cdots n-1,\ I(n\Delta t) \\ &= i\,|\,I(0)=i\}=q_{ii}^{(n)}-\sum_{m=1}^{n-1} q_{ii}^{(m)}f_{ii}^{(n-m)}(s) \end{aligned}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(34)$$

Next, the expected value of first passage time from i to j is represented by

$$\mu_{ij}(s)=\sum_{m=1}^{\infty}(m\Delta t)f_{ij}^{(m)}(s) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(35)$$

When the final state is i in Eq. (35), $\mu_{ii}(s)$ indicates the mean reoccurrence time from the level i to i. Finally, the first passage probability in the Markov process will be introduced by the following equation, which means the probability of transition from i to j for at least one time.

$$\begin{aligned} F_{ij}^{(\infty)}(s) &= P\{I(k\Delta t)=j \text{ for at least one } k \\ &\geq 1\,|\,I(0)=i\}=\sum_{m=1}^{\infty} f_{ij}^{(m)}(s)\,. \quad \cdots\cdots(36) \end{aligned}$$

When $j=i$, $F_{ii}^{(\infty)}(s)$ is called the first return probability.

In applying the above-mentioned relations to anticipate the earthquake occurrence, it is necessary to understand them precisely and to modify them more suitable forms.

When the transition probability q_{ij} represents the limiting probability in ergodic case, elements of transition matrix $\Pi(n\Delta t)$ after $t=n\Delta t$ will take the form,

$$q_{ij}(n\Delta t)=q_{ij}^{(n)}=q_{ij}(\Delta t)=q_j=\lambda_j\Delta t\,. \quad \cdots\cdots\cdots(37)$$

in which λ_j is the arrival rate of earthquake having the level j. Then, the first return time probability in Eq. (34) will be represented by

$$f_{ii}^{(n)}(s)=q_i\left(1-\sum_{m=1}^{n-1} f_{ii}^{(m)}(s)\right)\,. \quad \cdots\cdots\cdots\cdots\cdots\cdots(38)$$

By summing up Eq. (38) on m, the probability of occurrence of level i having more than one time during $n\Delta t$ takes the form

$$F_{ii}^{(n)}(s)=\sum_{m=1}^{n} f_{ii}^{(m)}(s)\,. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(39)$$

Eq. (38) will be transformed by use of Eq. (39) as follows.

$$\begin{aligned} f_{ij}^{(n)}(s) &= F_{ii}^{(n)}(s)-F_{ii}^{(n-1)}(s) \\ &= q_i(1-F_{ii}^{(n-1)}(s))\,. \quad \cdots\cdots\cdots\cdots\cdots\cdots(40) \end{aligned}$$

The regressive difference equation in Eq. (40) may be solved with regard to $F_{ii}^{(n)}(s)$.

$$\begin{aligned} F_{ii}^{(n)}(s) &= \sum_{m=1}^{n} f_{ii}^{(m)}(s)=q_i\sum_{m=0}^{n-1}(1-q_i)^m \\ &= \lambda_i\Delta t\sum_{m=0}^{n-1}(1-\lambda_i\Delta t)^m\,. \end{aligned}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(41)$$

By use of algebraic formula

$$\sum_{m=1}^{n} X^{m-1}=\frac{1-X^n}{1-X}\,. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(42)$$

Eq. (41) can be transformed to

$$\begin{aligned} F_{ii}^{(n)}(s) &= \frac{\lambda_i\Delta t\{1-(1-\lambda_i\Delta t)^n\}}{1-(1-\lambda_i\Delta t)} \\ &= 1-(1-\lambda_i\Delta t)^n\,. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(43) \end{aligned}$$

For $n \to \infty$, Eq. (43) yields the first return probability shown in Eq. (36), which tends to 1 in $\lambda_i \Delta t < 1$.

$$F_{ii}^{(\infty)}(s) = \lim_{n\to\infty} F_{ii}^{(n)}(s) = \lim_{n\to\infty}\{1-(1-\lambda_i \Delta t)^n\} = 1 \; . \qquad \cdots\cdots(44)$$

Substituting Eq. (43) into Eq. (40), the first return time probability is given by

$$f_{ii}^{(n)}(s) = \lambda_i \Delta t(1-\lambda_i \Delta t)^{n-1}. \qquad \cdots\cdots(45)$$

The mean reccurrence time $\mu_{ii}(s)$ in this case is no more than the mean arrival time in level i, $1/\lambda_i$. It is verified by setting $j=i$ in Eq. (35) and by introducing Eq. (45) into Eq. (35).

$$\mu_{ii}(s) = \sum_{m=1}^{\infty}(m\Delta t) f_{ii}^{(m)}(s) = \sum_{m=1}^{\infty}(m\Delta t)\lambda_i \Delta t(1-\lambda_i \Delta t)^{m-1}. \qquad \cdots\cdots(46)$$

By using algebraic formula

$$\sum_{m=1}^{\infty} m X^{m-1} = \frac{1}{(1-X)^2} \; . \qquad \cdots\cdots(47)$$

Eq. (46) will be transformed to

$$\mu_{ii}(s) = \frac{\lambda_i \Delta t^2}{\{1-(1-\lambda_i \Delta t)\}^2} = \frac{1}{\lambda_i} \; . \qquad \cdots\cdots(48)$$

As discussed in Section 3, it is more important for probabilistic approach in earthquake occurrence to formulate the passage state from the level i to j having no expreience above j before it reaches at its final level j. Otherwise, it has no meaning to apply various relations mentioned above in earthquake engineering. It will be shown in **Fig. 5** (a) and (b). For instance, the transition from i to j shown in **Fig. 5** (a) has no significance in earthquake engineering, because it is impossible to anticipate the "next" big earthquake occurrence. Since one of our purpose in structural design is to make structures so as to be in use during the structural life time previously designed, it is necessary to estimate the reoccurrence time (or passage time) for the biggest earthquake that the structure can resist. Therefore, as shown in **Fig. 5** (b), the reoccurrence time (or passage time) in this case should represent the transition from level i to i (or j) without having experience above i (or j) during $n\Delta t$.

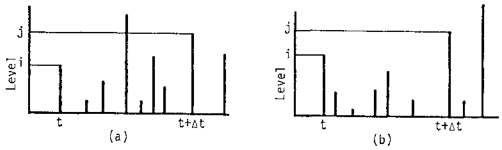

Fig. 5 Two Types of Transition from i to j.

The first passage time probability in this case can be derived by reference to Eq. (32).

$$P\{I(k\Delta t)<j,\, k=1,2,\cdots n-1,\, j\geq i,\, I(n\Delta t)=j \mid I(0)=i\}$$
$$= f_{ij}^{(n)}(j) = \sum_{a\neq j}^{j}\sum_{b\neq j}^{j}\cdots\cdots\sum_{u\neq j}^{j}\sum_{v\neq j}^{j} q_{ia}q_{ab}\cdots\cdots q_{uv}q_{vj}$$
$$= \sum_{a=0}^{j-1}\sum_{b=0}^{j-1}\cdots\cdots\sum_{u=0}^{j-1}\sum_{v=0}^{j-1} q_{ia}q_{ab}\cdots\cdots q_{uv}q_{vj}$$
$$= \sum_{a=0}^{j-1}\sum_{v=0}^{j-1} q_{ia}q_{av}^{(n-2)}(j)q_{vj} \; , \qquad \cdots\cdots(49)$$

in which j of $q_{av}^{(n-2)}(j)$ means the size of $j\times j$ matrix composed of the j raws $(0, 1, \cdots j-1)$ and j columns $(0, 1, \cdots j-1)$ in Eq. (16). That is,

$$q_{av}^{(n-2)}(j) = \begin{bmatrix} q_{00}, & q_{01}, & \cdots\cdots\cdots & q_{0,\,j-1} \\ q_{10}, & q_{11}, & \cdots\cdots\cdots & q_{1,\,j-1} \\ \cdots & \cdots & \cdots & \cdots \\ q_{j-1,\,0}, & & \cdots\cdots\cdots & q_{j-1,\,j-1} \end{bmatrix}^{(n-2)} . \qquad \cdots\cdots(50)$$

Eq. (49) indicates the transition probability from the level i to j $(j\geq i)$ having no experience above j before it reaches at its final level j in $t=n\Delta t$. Then, the first return time probability at $t=n\Delta t$ will take the form

$$f_{ii}^{(n)}(i) = \sum_{a=0}^{i-1}\sum_{v=0}^{i-1} q_{ia}q_{av}^{(n-2)}(i)q_{vi} \; . \qquad \cdots\cdots(51)$$

The states described above have only one final level, which might restrict their application to the model in earthquake occurrence. For instance, if a structure is designed against level i, it means that the structure cannot resist the earthquake above level i. Therefore, it is necessary to know the reoccurrence probability and reccurrence time against the earthquakes above level i. In this case it is sufficient to replace q_{vi} in Eq. (51) by

$$R_{vI} = P\{I(\Delta t)\geq i \mid I(0)=v\} = \sum_{k=i}^{s-1} q_{vk} \; . \qquad \cdots\cdots(52)$$

Then,

$$f_{iI}^{(n)}(i) = P\{I(k\Delta t)<i,\, k=1,2,\cdots n-1,\, I(n\Delta t) \geq i \mid I(0)=i\} = \sum_{a=0}^{i-1}\sum_{v=0}^{i-1} q_{ia}q_{av}^{(n-2)}(i)R_{vI} \; . \qquad \cdots\cdots(53)$$

Eq. (53) is the probability of occurrence above i for the first time at $t=n\Delta t$ which means, in other words, the probability that the earthquake above level i never occurs from $t=\Delta t$ to $t=(n-1)\Delta t$ and occurs for the first time at $t=n\Delta t$. It will be applicable to estimate the probability of structural safety against earthquakes. In this case the expected value of first return time is represented by

$$\mu_{Iv}(i)=\sum_{m=1}^{\infty}(m\Delta t)f_{iI}^{(m)}(i) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(54)$$

It is the expected mean reoccurrence time above level i. Similarly, in Eq. (49) q_{vj} is replaced by R_{vj} to obtain the first passage time probability and the expected values of first passage time above level j.

When $q_{ij}=q_j=\lambda_j\Delta t$ in ergodic case, these equations described above will be reduced to the more simple forms. The transition probabilities given by Eq. (50) is transformed by substituting $q_{ij}=\lambda_j\Delta t$.

$$q_{av}^{(n-2)}(i)=(1-\lambda_I\Delta t)^{n-3}\begin{bmatrix}\lambda_0\Delta t, & \cdots\cdots & \lambda_{i-1}\Delta t\\ \vdots & & \vdots\\ \lambda_0\Delta t, & \cdots\cdots & \lambda_{i-1}\Delta t\end{bmatrix} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(55)$$

By introducing Eq. (55) into Eq. (51), the first return time probability at $t=n\Delta t$ is represented by the form

$$f_{ii}^{(n)}(i)=\lambda_i\Delta t(1-\lambda_I\Delta t)^{n-1} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(56)$$

Similarly, Eq. (53) and Eq. (54) are transformed to

$$f_{iI}^{(n)}(i)=\lambda_I\Delta t(1-\lambda_I\Delta t)^{n-1} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(57)$$

$$\mu_{iI}(i)=\lambda_I\Delta t^2\sum_{m=1}^{\infty}m(1-\lambda_I\Delta t)^{m-1} \quad \cdots\cdots\cdots\cdots(58)$$

Eq. (58) will be finally transformed to the simplest forms by using the relation of Eq. (47)

$$\mu_{iI}(i)=\frac{\lambda_I\Delta t^2}{\{1-(1-\lambda_I\Delta t)\}^2}=\frac{1}{\lambda_I} \quad \cdots\cdots\cdots\cdots(59)$$

This relation is clearly comparable to Eq. (48). Eq. (59) is nothing but the return period of earthquakes greater than level i in ergodic case.

The probability of exceedence above level i during $n\Delta t$ will be represented by use of Eq. (42).

$$F_{iI}^{(n)}(i)=\sum_{m=1}^{n}f_{iI}^{(m)}(i)=\lambda_I\Delta t\sum_{m=1}^{n}(1-\lambda_I\Delta t)^{m-1}$$
$$=1-(1-\lambda_I\Delta t)^n \quad \cdots\cdots(60)$$

It is naturally the same equation as Eq. (13) and (30).

Fig. 6 illustrates three types of first return time probabilities calculated by the data shown in **Table** 2, in which solid lines indicate $f_{iI}^{(n)}(i)$ of Eq. (57), broken lines, $f_{ii}^{(n)}(i)$ of Eq. (56) and chained lines, $f_{ii}^{(n)}(s)$ of Eq. (45). In these curves $f_{ii}^{(n)}(i)$ comes closer to $f_{iI}^{(n)}(i)$ as time passes and $f_{ii}^{(n)}(s)$ converges to the slightly higher values than $f_{iI}^{(n)}(i)$. As mentioned before, only $f_{iI}^{(n)}(i)$ of solid lines, which means the probability that the earthquake above intensity level i will recur for the first time at $t=n\Delta t$, has significant meaning in earthquake engineering.

It is clear that $f_{iI}^{(n)}(i)$ of intensity level III, IV and V will converge after 2 years, 10 years and 50 years, respectively. It means that after 2 years, 10 years and 50 years the earthquakes above intensity level III, IV and V will not probably recur in Tokyo area. The earthquake above intensity level V, for instance, will recur for the first time after 5 years with the probability 0.005 in a month but hardly recur with 50 years' interval.

The period at the intersecting point of solid line suggests the safety index of structural life design. If one wants to design such a structures that its structural life is 25 years, he may design

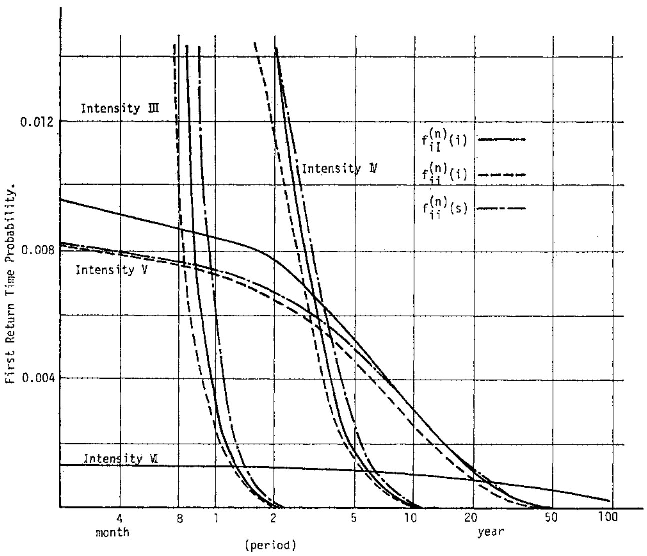

Fig. 6 First Return Time Probability of Earthquake.

Table 4 Safety Period of Structural Life Compared with the Intensity Level of Earthquake.

Intensity Scale (JMA)	III	IV	V	VI
Safety Period	— 6 months	6 months — 3.2 years	3.2 years — 25 years	25 years —

it safely against the earthquake having intensity V, because the first return probability of earthquake occurrence above level V is greater than that of intensity level VI during 25 years from now. But if one wants to design the structure of which structural life is beyond 25 years, he has to design it against the earthquake above intensity level VI. **Table 4** shows the safety periods of structural life obtained by **Fig. 6**. It is interesting to compare it with the economical periods shown in **Table 3**. It should be noted that the application of these result should be combined with the importancy of structures, the construction and depreciation cost of structures and so on.

5. CONCLUSION

The authors discussed three types of discontinuous Markov processes and their application on probabilistic approaches to the earthquake occurrence in Tokyo. The purpose of this paper is firstly to clarify the relationship between these three models, and secondly to represent how to apply them in probabilistic prediction of future earthquake occurrence. If the transition matrix of earthquake occurrence would not be homogenious and stationary, this model might be more effective and useful to estimate the accurate relation between structural life and seismic loading. Though there are many other models of discontinuous Markov process considered, it will be very important to understand precisely the relation between them.

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