

LINEARIZATION TECHNIQUES FOR EARTHQUAKE RESPONSE OF SIMPLE HYSTERETIC STRUCTURES

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1. INTRODUCTION

In recent years, statistical aspects of structural vibration induced by earthquake excitation have received considerable attention of many researchers. A majority of these statistical works, however, has only dealt with structures with linear restoring force¹⁾.

To evaluate the reliability of structures during earthquakes, it is considered essential to investigate statistical characteristics of the response of structures with hysteretic restoring forces, since most structures show weak or strong yielding behavior in strong earthquakes. However, from the reason that the principle of superposition (Duhamel's integral) cannot be applied to nonlinear structures, it is generally very difficult to make theoretical analysis of their earthquake response.

As an exact analytical method, we have the Fokker-Planck equation²⁾. But at the present stage, it is applicable only to the stationary response subjected to white noise excitation³⁾. It is also of great significance in the investigation of earthquake response to consider the frequency characteristics and the nonstationarity of the excitation, since most of the strong earthquake motions have the predominant frequency⁴⁾ and the structures start to vibrate from the static state. We cannot discuss these important characteristics of the excitation and of the response by the solution of the Fokker-Planck equation at the present stage.

On the other hand, the numerical methods such as a step by step integration of the equation of motion on a digital computer have a great applicability for any kind of hysteresis and for any kind of excitation⁵⁾. However these methods

consume a lot of time for the calculation of many samples. Moreover we have to consider additional ruling parameters due to hysteretic characteristics of the structures. Therefore, accumulation of a great amount of samples seems to be needed for each parameters to make theoretical statements about the earthquake response characteristics of nonlinear hysteretic structures by the numerical methods.

To overcome these difficulties, equivalent linearization techniques are very powerful methods in the range of admissible error, since the results of random vibration theory in linear structures can be available⁶⁾.

In this paper, two equivalent linearization techniques were used to find the general properties of earthquake response of nonlinear hysteretic structures. By using these two methods, not only the stationary response but also the nonstationary response of structures with hysteresis were predicted. Moreover numerical simulations were performed on a digital computer to seek for the applicability of the equivalent linearization techniques used herein.

2. EQUIVALENT LINEARIZATION TECHNIQUES

To make theoretical discussions about the earthquake response characteristics of nonlinear hysteretic structures more general, a dimensionless representation of the equation of motion was tried in the previous paper⁷⁾, which leads to the following equation of motion of a single-degree-of-freedom structure with viscous damping and with any type of nonlinear hysteretic characteristics:

$$\begin{aligned} \ddot{\mu}(t) + \beta_0 \dot{\mu}(t) + q(\alpha, \beta, \mu, \dot{\mu}, t) \\ = -r_s \phi(t) f(\eta, h_f, t) \dots\dots\dots (1) \end{aligned}$$

where $\mu(t)$: ductility factor, α and β : parameters which show characteristics of dimensionless hysteresis $q(\alpha, \beta, \mu, \dot{\mu}, t)$, β_0 : damping coefficient in small oscillation of yielding structures, \cdot : deriv-

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ative by dimensionless time, t , r_s : a constant showing the intensity of the excitation, $\phi(t)$: a deterministic shape function which exclusively assumes positive values, $f(\eta, h_f, t)$: a stationary non-white random process with zero mean value and the variance of unity, $\eta = \omega_f/\omega_0$ and h_f : parameters showing the characteristics of the power spectrum of $f(t)$, ω_f : predominant frequency of $f(t)$, ω_0 ($=1.0$): natural frequency of small oscillation of yielding structure, respectively.

By using the equivalent damping coefficient β_{eq} and the equivalent natural frequency ω_{eq} , the equation of motion of the equivalent linear structure can be written as follows:

$$\ddot{u}(t) + \beta_{eq}\dot{u}(t) + \omega_{eq}^2 u(t) = -r_s \phi(t) f(\eta, h_f, t) \dots\dots\dots(2)$$

Concerning the techniques to determine β_{eq} and ω_{eq} both in sinusoidal and in random vibration, discussions have been made in the previous paper⁸). After a further investigation of these techniques, an interesting result about them has been found. In connection with it, the following discussions shall be made as to two typical linearization techniques; one is the least mean-square error method first developed by T. K. Caughey⁹), and the other is what we may call the energy balance method first proposed by L. S. Jacobsen¹⁰). These two methods have been discussed separately and their relation with each other has never been investigated. However, we shall see in the following sections that they are closely correlated to each other.

(1) The Least Mean-Square Error Method

In this method, the two equivalent linearization parameters β_{eq} and ω_{eq} are determined so as to minimize the mean-square error between Eqs. (1) and (2). The mean-square error in one cycle ($\tau, \tau + \tau_T$) can be written as follows:

$$I(\beta_{eq}, \omega_{eq}) = \frac{1}{\tau_T} \int_{\tau}^{\tau + \tau_T} \{ \beta_0 \dot{u} + q - \beta_{eq} \dot{u} - \omega_{eq}^2 u \}^2 dt \dots\dots\dots(3)$$

Now, let us minimize $I(\beta_{eq}, \omega_{eq})$ with respect to β_{eq} and ω_{eq}^2 .

$$\left. \begin{aligned} \frac{\partial I}{\partial \beta_{eq}} &= \beta_0 \oint \dot{u}^2 dt + \oint q \dot{u} dt - \beta_{eq} \oint \dot{u}^2 dt \\ &\quad - \omega_{eq}^2 \oint \mu \dot{u} dt = 0 \\ \frac{\partial I}{\partial \omega_{eq}^2} &= \beta_0 \oint \mu \dot{u} dt + \oint q \mu dt - \beta_{eq} \oint \mu \dot{u} dt \\ &\quad - \omega_{eq}^2 \oint \mu^2 dt = 0 \end{aligned} \right\} \dots\dots\dots(4)$$

in which $\oint dt$ denotes the integration over one cycle of oscillation. If the nonlinearity of hysteresis is weak and the damping is slight, we can assume that the response $\mu(t)$ is a sinusoidal time function with a slowly varying random amplitude $\mu_0(t)$ and a random phase angle $\phi(t)$; i.e.,

$$\mu(t) = \mu_0(t) \cos \{ \omega_{eq} t + \phi(t) \} \dots\dots\dots(5)$$

By substituting Eq. (5) to Eq. (4), we obtain

$$\left. \begin{aligned} \beta_{eq}(\mu_0) &= \beta_0 + \frac{\oint q \dot{\mu} dt}{\oint \dot{\mu}^2 dt} \\ \omega_{eq}^2(\mu_0) &= \frac{\oint q \mu dt}{\oint \mu^2 dt} \end{aligned} \right\} \dots\dots\dots(6)$$

(2) Energy Balance Method

In this method, the equivalent damping coefficient is determined so as to equate the energy dissipation by the hysteretic structure to that of the linear structure; i.e.,

$$\oint \{ \beta_0 \dot{\mu} + q \} d\mu = \oint \beta_{eq} \dot{\mu} d\mu \dots\dots\dots(7)$$

The equivalent natural frequency can be determined independently by various methods. But it seems to be reasonable to obtain it as the resonance frequency of structures with hysteresis. In the previous study⁸), the resonance curve was obtained as

$$\{ S(\mu_0) - \beta_0 \mu_0 \omega \}^2 + \{ C(\mu_0) - \mu_0 \omega^2 \}^2 = r_s^2 \dots\dots\dots(8)$$

where

$$\left. \begin{aligned} S(\mu_0) &= \frac{1}{\pi} \int_0^{2\pi} q(\alpha, \beta, \mu_0 \cos \theta) \sin \theta d\theta \\ C(\mu_0) &= \frac{1}{\pi} \int_0^{2\pi} q(\alpha, \beta, \mu_0 \cos \theta) \cos \theta d\theta \end{aligned} \right\} \dots\dots\dots(9)$$

The resonance frequency is determined by letting

$$\frac{\partial \mu_0(\omega)}{\partial \omega} = 0 \dots\dots\dots(10)$$

From Eqs. (7)~(10), the equivalent linearization parameters can be obtained as follows:

$$\left. \begin{aligned} \beta_{eq}(\mu_0) &= \beta_0 + \frac{\oint q d\mu}{\oint \dot{\mu} d\mu} \\ \omega_{eq}^2(\mu_0) &= \frac{1}{\mu_0} C(\mu_0) \end{aligned} \right\} \dots\dots\dots(11)$$

Under the assumption of Eq. (5) and considering that $d\mu = \dot{\mu} dt$, the results of Eqs. (6) and (11) completely coincide with each other for any type of hysteresis. It is considered very interesting from the physical viewpoint that the quite different methods conclude the same expressions of the equivalent linear damping coefficient and natural

frequency. That is to say, the equivalent linear structure determined from the least mean-square error method has the same resonance frequency and moreover dissipates the same amount of energy as that of hysteretic structures. Therefore we can conclude that the least mean-square error method is physically well-grounded in linearizing the structures with hysteresis.

As a typical example of dimensionless hysteretic characteristics, bilinear hysteresis shown in Fig. 1 is considered in this study. The yielding

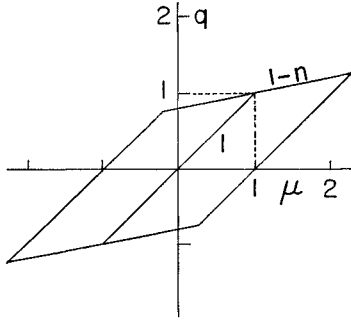


Fig. 1 Nondimensional representation of bilinear hysteresis.

point is defined as the point at which $\mu=q=1$. The initial stiffness before yielding is unity and the second stiffness after yielding is $(1-n)$, where n is the parameter which shows the nonlinearity of the bilinear hysteresis. So that this hysteresis shows the linear structure if $n=0$, and the perfect elasto-plastic structure if $n=1$. Then after some algebraic treatments, expressions in Eqs. (6) and (11) for the bilinear hysteresis yield

$$\left. \begin{aligned}
 \mu_0 \leq 1: \quad \beta_{eq}(\mu_0) &= \beta_0 \\
 \mu_0 > 1: \quad \beta_{eq}(\mu_0) &= \beta_0 + \frac{4n(\mu_0 - 1)}{\pi \omega_{eq}^2 \mu_0^2} \\
 \mu_0 \leq 1: \quad \omega_{eq}^2(\mu_0) &= 1 \\
 \mu_0 > 1: \quad \omega_{eq}^2(\mu_0) &= \frac{2n(2 - \mu_0)}{\pi \mu_0^2} \sqrt{\mu_0 - 1} \\
 &\quad + \frac{n}{\pi} \cos^{-1} \left(1 - \frac{2}{\mu_0} \right) + (1 - n)
 \end{aligned} \right\} \dots\dots\dots(12)$$

Numerical values for Eq. (12) are shown in Fig. 2 for the parameters $n=0.25, 0.50, 0.75, 1.00$. The values of the equivalent damping coefficient $\beta_{eq}(\mu_0)$ and the equivalent natural frequency $\omega_{eq}(\mu_0)$ within the elastic region ($\mu_0 \leq 1.0$) are β_0 and unity, respectively. Since the area of the bilinear hysteresis loop is proportional to n which shows the nonlinearity, the value of the equivalent damping coefficient becomes greater according to the

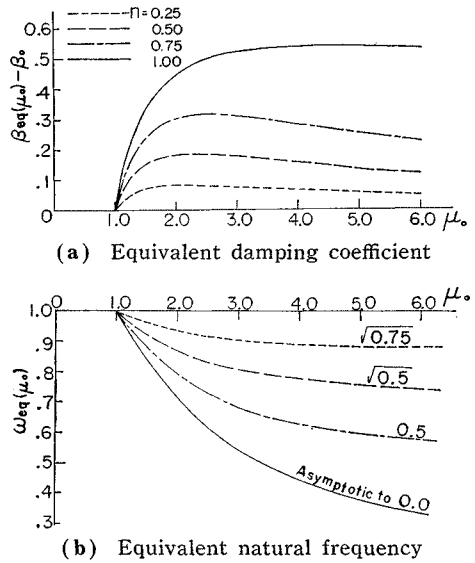


Fig. 2 Equivalent linearization parameters.

increase in the parameter n and it is asymptotic to zero when the amplitude increases infinitely. The equivalent natural frequency shows smaller value with the increase of n . In the limit of infinite amplitude, it is asymptotic to $\sqrt{1-n}$ in which $(1-n)$ is the second spring constant after yielding.

(3) Equivalent Linearization Parameters in Random Response

From the analysis in the previous section, the equivalent damping coefficient and the equivalent natural frequency are obtained as a function of response amplitude μ_0 . Direct application of these results to random response is impossible. But if the probability distribution $P(\mu_0)$ of the peak amplitude μ_0 in random response can be estimated, the equivalent linearization parameters in random response would be defined as their expectations using $P(\mu_0)$. Thus in the case of stationary random response, the equivalent linearization parameters would be defined as a function of standard deviation σ_μ of μ in the following form:

$$\left. \begin{aligned}
 \beta_{eq}(\sigma_\mu) &= \int_0^\infty \beta_{eq}(\mu_0) P(\mu_0, \sigma_\mu) d\mu_0 \\
 \omega_{eq}^2(\sigma_\mu) &= \int_0^\infty \omega_{eq}^2(\mu_0) P(\mu_0, \sigma_\mu) d\mu_0
 \end{aligned} \right\} \dots\dots\dots(13)$$

in which the peak amplitude distribution has been obtained by S. O. Rice²²⁾ as

$$P(\mu_0, \sigma_\mu) = \frac{\mu_0}{\sigma_\mu^2} \exp\left(-\frac{\mu_0^2}{2\sigma_\mu^2}\right) \dots\dots\dots(14)$$

In the case of nonstationary random response, the

peak amplitude distribution which has been obtained by T. Kobori and R. Minai¹³⁾ as a function of standard deviation of μ and $\dot{\mu}$ and the correlation coefficient between them shall be used in this study. Then we obtain

$$\beta_{eq}(\sigma_\mu, \rho_{\mu\dot{\mu}}, \sigma_{\dot{\mu}}) = \int_0^\infty \beta_{eq}(\mu_0) P(\mu_0, \sigma_\mu, \rho_{\mu\dot{\mu}}, \sigma_{\dot{\mu}}) d\mu_0$$

$$\omega_{eq}^2(\sigma_\mu, \rho_{\mu\dot{\mu}}, \sigma_{\dot{\mu}}) = \int_0^\infty \omega_{eq}^2(\mu_0) P(\mu_0, \sigma_\mu, \rho_{\mu\dot{\mu}}, \sigma_{\dot{\mu}}) d\mu_0$$

where,

$$P(\mu_0, \sigma_\mu, \rho_{\mu\dot{\mu}}, \sigma_{\dot{\mu}}) = \exp\left(-\frac{\mu_0^2}{2\sigma_\mu^2}\right) \left[\frac{\mu_0}{\sigma_{\dot{\mu}}} \exp\left(-\frac{\rho\mu_0^2}{2(1-\rho^2)\sigma_\mu^2}\right) + \frac{\rho}{\sqrt{1-\rho^2}} \sqrt{\frac{\pi}{2\sigma_\mu^2}} \left(\frac{\mu_0^2}{\sigma_\mu^2} - 1\right) \cdot \operatorname{erf}\left(\frac{\rho\mu_0}{\sqrt{2(1-\rho^2)\sigma_\mu^2}}\right) \right]$$

$$\rho = \frac{\sigma_{\mu\dot{\mu}}}{(\sigma_\mu \sigma_{\dot{\mu}})} \dots\dots\dots(15)$$

3. STATIONARY RESPONSE

(1) Iterative Method

In this chapter, we will investigate the stationary response of structures with hysteresis subjected to stationary random excitation $r_s f(t)$ both by analytical and by experimental methods.

In the previous chapter, equivalent linearization parameters have been determined analytically as a function of the r.m.s. response. But the stationary r.m.s. response itself is still unknown. From the experimental methods such as numerical simulations on digital or analog computers, we would be able to obtain it. However, analytical methods to predict the response of nonlinear structures are considered to be much more important in order to make theoretical discussions about the random response of these structures.

It is also desirable in earthquake response analysis to consider the frequency characteristics of the excitation. In this study, therefore, we take as the stationary excitation $f(t)$ of which power spectrum shall be represented in the following form analogous to the receptance of the relative velocity of a simple structure:

$$S_f(\omega) = \frac{4h_f}{\pi\omega_f} \frac{(\omega/\omega_f)^2}{\{1 - (\omega/\omega_f)^2\}^2 + 4h_f^2(\omega/\omega_f)^2} \dots\dots\dots(16)$$

in which h_f is the damping factor and ω_f is the natural frequency of a simple structure. It is

readily verified that Eq. (16) satisfies the condition that $f(t)$ should have the variance of unity.

The stationary r.m.s. response of the equivalent linear structure of which parameters β_{eq} and ω_{eq} can be obtained from Eqs. (13) and (14) can easily be estimated by using the random vibration theory for linear structures as

$$\sigma_\mu^2 = \frac{1}{\pi} \int_0^\infty S_f(\omega) |H(\omega)|^2 d\omega$$

$$= \frac{1}{\pi} \int_0^\infty \frac{r_s^2 S_f(\omega) (\omega_f/\eta)^4}{(\omega_{eq}^2 - \omega^2)^2 + (\beta_{eq}\omega)^2} d\omega \dots\dots\dots(17)$$

Then the following three relations are obtained for the stationary random response of hysteretic structures from Eqs. (13) and (17).

$$\left. \begin{aligned} \beta_{eq} &= \text{func.}(\sigma_\mu) \\ \omega_{eq} &= \text{func.}(\sigma_\mu) \\ \sigma_\mu &= \text{func.}(\beta_{eq}, \omega_{eq}, r_s, \omega_f, h_f) \end{aligned} \right\} \dots\dots\dots(18)$$

By solving three relations, equivalent linear parameters β_{eq} and ω_{eq} would be determined as those of the optimum equivalent linear structure and the r.m.s. response of it would also be predicted. However it is quite hard to obtain the solution of Eq. (18) explicitly. Therefore, to find the numerical results of Eq. (18), an iterative method was used on a digital computer¹¹⁾. In this method, we first estimate the r.m.s. response of the linear structure with parameters ω_0 and β_0 , then we obtain corresponding equivalent linear parameters ω_{eq} and β_{eq} from Eq. (13). By substituting these parameters to Eq. (17), we have the r.m.s. response σ_μ . Repeating this procedure, the response approaches a constant value.

(2) Calculated Results

The numerical results for the stationary r.m.s. response of structures with bilinear hysteresis and corresponding equivalent linear parameters are shown in Fig. 3 for the set of parameters $r_s=0.5$, $\beta_0=0.1$ and $n=0.00, 0.25, 0.50, 0.75$.

In Fig. 3(a), we can find that the r.m.s. response of the bilinear structure with a stronger nonlinearity of hysteresis is comparatively less than that of the linear structure, $n=0.00$, that is $\omega_{eq}=\omega_0$ and $\beta_{eq}=\beta_0$ in the frequency range from $\eta=\omega_f/\omega_0=0.75$ to 3.0. On the contrary, in the frequency range $\eta \leq 0.75$ the r.m.s. response is not necessarily smaller than that of the linear structure.

In order to discuss these response characteristics of hysteretic structures generally, the variation of ω_{eq} and β_{eq} which are shown in Fig. 3(b) and (c) should be investigated. The equivalent natural frequency ω_{eq} is always less than ω_0 be-

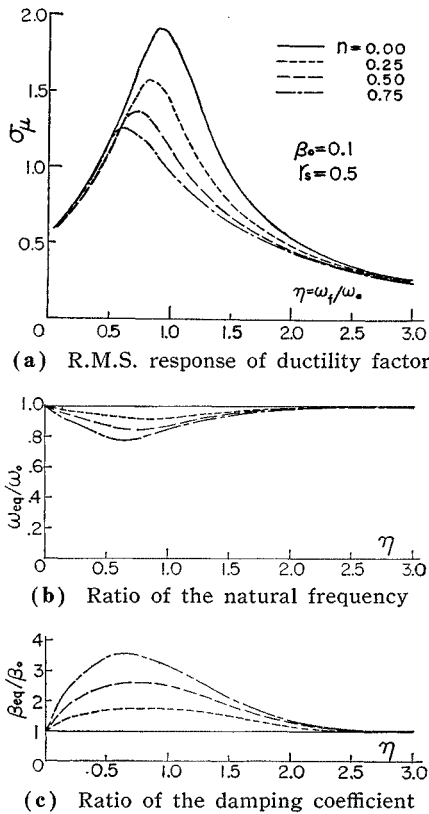


Fig. 3 Stationary response of structures with bilinear hysteresis.

cause of the softening-type spring characteristics, and the greater n is, the less ω_{eq} becomes. The equivalent damping coefficient β_{eq} is necessarily greater than β_0 in consequence of hysteretic energy dissipation: it increases with increasing n .

The effect of such properties of ω_{eq} and β_{eq} in the stationary r.m.s. response would reasonably be explained from the concept of the transition of the receptance of the structure due to its hysteretic properties schematically illustrated in Fig. 4 in terms of the spectrum coordinate. That is to say, as the structure softens due to yielding and consequently ω_{eq} decreases, the receptance of a relatively "rigid" structure ($\eta=0.5$; $\omega_0 > \omega_f$) moves closer to the peak of the spectrum of the excitation and tends to increase the response. On the contrary, such a shift of the receptance tends to suppress the response of a relatively "soft" structure ($\eta=2.0$; $\omega_0 < \omega_f$). Besides, an increase in β_{eq} limits the response over the whole frequency range, and the compound effect of β_{eq} and ω_{eq} results in Fig. 3(a). Thus the concept of the transition of receptance seems very powerful to

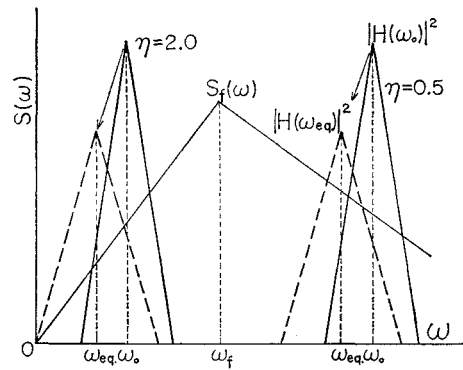


Fig. 4 Transition of the receptance due to nonlinearity and hysteresis.

make theoretical discussions about the random response characteristics of structures with any kind of hysteresis.

(3) Numerical Simulation

In order to check the accuracy of the prediction of the r.m.s. response in the previous section, a numerical simulation has been carried out on a digital computer.

The stationary artificial earthquake has been generated by the following procedure. First, a white noise was generated as the summation of 500 sinusoidal time functions of which circular frequency and phase angle are random variables with uniform probability densities¹⁴). Then the velocity response of a simple structure with parameters ω_f and h_f subjected to this white noise was calculated by the linear acceleration method. The stationary part of the velocity was taken as the excitation $f(t)$.

The response of the structures with bilinear hysteresis was obtained by integrating the governing equation of motion by the linear acceleration method. The r.m.s. response σ_μ of the ductility factor μ was calculated as the time average over the stationary duration of 20 times the natural period of infinitesimal vibration.

Both analytical and experimental results are shown in Fig. 5 for the same sets of parameters as in Fig. 3. It is observed that the analytical and simulated results agree rather well for the parameters of $n=0.25, 0.50$ and 0.75 as shown in Figs. 5(a), (b) and (c). So within the limits of these parameters, it could be said that the iterative method investigated in the previous section is very effective to predict the r.m.s. response of the bilinear structures.

Concerning the distribution functions of the

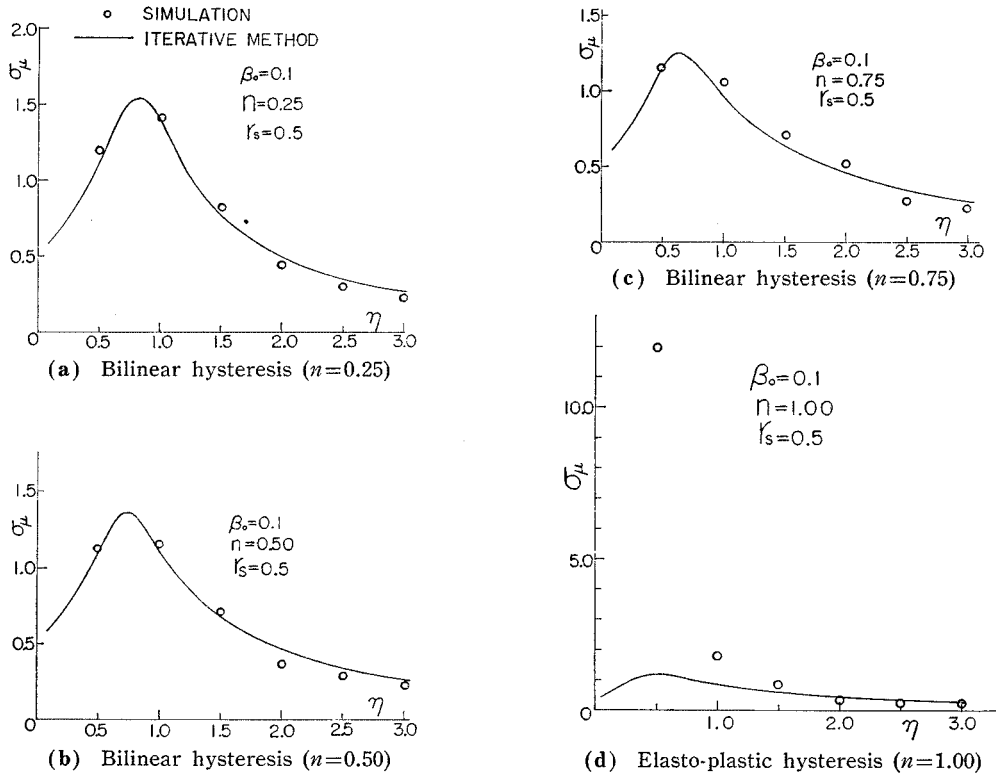


Fig. 5 Comparison between simulated and predicted results of stationary response.

ductility factor μ of these structures, H. Kameda and one of the authors showed in their previous paper¹¹⁾ that they are not on the whole extremely different from the Gaussian distribution. Therefore the statistical properties of the stationary response of these structures could be explained by using the Gaussian distribution with zero mean and the standard deviation predicted by the equivalent linearization used herein.

On the other hand, the simulated result shows much greater values than the analytical result for the structure with perfect elasto-plastic hysteresis ($n=1.0$). This discrepancy is remarkable especially for a "rigid" structure ($\eta=0.5$). For a "soft" structure, it is not remarkable since the response remains almost in the elastic region.

These effects are mainly attributable to the growth of the plastic deformation due to an excessive yielding behavior. The authors pointed out by using the moving average method that a conspicuous plastic deformation occurs only at the structure with perfect elasto-plastic hysteresis⁷⁾. Therefore the r.m.s. response cannot be predicted by the equivalent linearization techniques without estimating the plastic deformation

analytically.

4. NONSTATIONARY RESPONSE

(1) Step-by-Step Method

In the previous chapter, the equivalent linear structures and their response in the stationary state were investigated. It is, however, very important in the reliability analysis of structures in strong earthquakes to investigate the nonstationarity of response of structures with hysteresis for the following reasons. 1) The strength of earthquake motions varies with time. This time variation seems to depend on the location of the observation site relative to the hypocenter and on the pass characteristics, etc. 2) Even we might simply assume that the earthquake excitation is stationary, structures in earthquakes start to vibrate from the static state.

In this section, we shall discuss the step-by-step method^{13), 15)} to obtain the variances of displacement and velocity response, and the correlation coefficient between them at each step of time.

Under the initial conditions that

$$\begin{aligned} \mu(t)|_{t=0} &= \mu_0, & \dot{\mu}(t)|_{t=0} &= \dot{\mu}_0, \\ \psi(t)|_{t=0} &= \psi_0 \end{aligned} \quad \dots\dots\dots(19)$$

the solution of Eq. (2) is obtained by the summation of the free vibration due to initial condition and the forced vibration due to excitation as follows:

$$\left. \begin{aligned} \mu(t) &= -\frac{r_s \psi_0}{p} \int_0^t h(t-t') f(t') dt' + I(t) \\ \dot{\mu}(t) &= -\frac{r_s \dot{\psi}_0}{p} \int_0^t \dot{h}(t-t') f(t') dt' + \dot{I}(t) \end{aligned} \right\} \dots\dots\dots(20)$$

where

$$\left. \begin{aligned} p &= \sqrt{\omega_{eq}^2 - \beta_{eq}^2/4}, \\ I(t) &= g_1(t)\mu_0 + g_2(t)\dot{\mu}_0, \\ \dot{I}(t) &= g_3(t)\mu_0 + g_4(t)\dot{\mu}_0 \\ h(t) &= \exp(-\beta_{eq}t/2) \sin pt, \\ \dot{h}(t) &= p \exp(-\beta_{eq}t/2) \\ &\quad \cdot \{\cos pt - (\beta_{eq}/2p) \sin pt\}, \\ g_1(t) &= \exp(-\beta_{eq}t/2) \{\cos pt + (\beta_{eq}/2p) \sin pt\}, \\ g_2(t) &= (1/p) \exp(-\beta_{eq}t/2) \sin pt, \\ g_3(t) &= -(\omega_{eq}^2/p) \exp(-\beta_{eq}t/2) \sin pt, \\ g_4(t) &= -(\beta_{eq}/2) \exp(-\beta_{eq}t/2) \sin pt \\ &\quad + \exp(-\beta_{eq}t/2) \cos pt \end{aligned} \right\} \dots\dots\dots(21)$$

From Eqs. (20) and (21), the variance of displacement and of velocity response and the correlation coefficient between them at $t=\xi_1$ are represented by

$$\left. \begin{aligned} \sigma_{\mu}^2(\xi_1) &= E[\mu^2(\xi_1)] \\ &= (r_s^2 \psi_0^2/p^2) \int_0^{\xi_1} \int_0^{\xi_1} h(\xi_1-t') h(\xi_1-t'') \\ &\quad \cdot E[f(t') f(t'')] dt' dt'' - (2r_s \psi_0/p) \\ &\quad \cdot E \left[I(\xi_1) \int_0^{\xi_1} h(\xi_1-t') f(t') dt' \right] \\ &\quad + E[I^2(\xi_1)] \\ \sigma_{\dot{\mu}}^2(\xi_1) &= E[\dot{\mu}^2(\xi_1)] \\ &= (r_s^2 \dot{\psi}_0^2/p^2) \int_0^{\xi_1} \int_0^{\xi_1} \dot{h}(\xi_1-t') \dot{h}(\xi_1-t'') \\ &\quad \cdot E[f(t') f(t'')] dt' dt'' - (2r_s \dot{\psi}_0/p) \\ &\quad \cdot E \left[\dot{I}(\xi_1) \int_0^{\xi_1} \dot{h}(\xi_1-t') f(t') dt' \right] \\ &\quad + E[\dot{I}^2(\xi_1)] \\ \rho_{\mu \dot{\mu}}(\xi_1) \cdot \sigma_{\mu} \cdot \sigma_{\dot{\mu}} &= E[\mu(\xi_1) \dot{\mu}(\xi_1)] \\ &= (r_s^2 \psi_0 \dot{\psi}_0/p^2) \int_0^{\xi_1} \int_0^{\xi_1} h(\xi_1-t') \dot{h}(\xi_1-t'') \\ &\quad \cdot E[f(t') f(t'')] dt' dt'' - (r_s \psi_0/p) \\ &\quad \cdot E \left[I(\xi_1) \int_0^{\xi_1} \dot{h}(\xi_1-t'') f(t'') dt'' \right] \\ &\quad - (r_s \dot{\psi}_0/p) E \left[\dot{I}(\xi_1) \int_0^{\xi_1} h(\xi_1-t') f(t') dt' \right] \\ &\quad + E[\dot{I}(\xi_1) I(\xi_1)] \end{aligned} \right\} \dots\dots\dots(22)$$

where

$$\left. \begin{aligned} E[I^2(\xi_1)] &= g_1^2(\xi_1) E[\mu_0^2] + 2g_1(\xi_1)g_2(\xi_1) E[\mu_0 \dot{\mu}_0] \\ &\quad + g_2^2(\xi_1) E[\dot{\mu}_0^2] \\ E[\dot{I}^2(\xi_1)] &= g_3^2(\xi_1) E[\mu_0^2] + 2g_3(\xi_1)g_4(\xi_1) E[\mu_0 \dot{\mu}_0] \\ &\quad + g_4^2(\xi_1) E[\dot{\mu}_0^2] \\ E[I(\xi_1) \dot{I}(\xi_1)] &= g_1(\xi_1)g_3(\xi_1) E[\mu_0^2] \\ &\quad + \{g_1(\xi_1)g_4(\xi_1) + g_2(\xi_1)g_3(\xi_1)\} \\ &\quad \cdot E[\mu_0 \dot{\mu}_0] + g_2(\xi_1)g_4(\xi_1) E[\dot{\mu}_0^2] \end{aligned} \right\} \dots\dots\dots(23)$$

The values at $t=\xi_1$ of the three terms on the left side of Eq. (22) can be obtained if those at $t=0$ are known. Hence the nonstationary response can be estimated by dividing the duration into many steps and applying Eq. (22) to each step.

The first terms of the right side of Eq. (22) were calculated by using the approximate results obtained by H. Kameda and one of the authors⁽⁶⁾ taking the same technique of T. K. Caughey and H. J. Stumph⁽⁷⁾.

The covariance between the free vibration and the forced vibration will become zero if the power spectrum density of the excitation is white, since

$$\begin{aligned} E \left[I(\xi_1) \int_0^{\xi_1} h(\xi_1-t') f(t') dt' \right] \\ &= E \left[\int_{-\xi_1}^0 h(o-t'') f(t'') dt'' \right. \\ &\quad \cdot \left. \int_0^{\xi_1} h(\xi_1-t') f(t') dt' \right] \\ &= \int_{-\xi_1}^0 \int_0^{\xi_1} h(o-t'') h(\xi_1-t') \\ &\quad \cdot E[f(t'') f(t')] dt'' dt' \\ &= \int_{-\xi_1}^0 \int_0^{\xi_1} h(o-t'') h(\xi_1-t') \delta(t' - t'') dt' dt'' \\ &= 0 \end{aligned} \quad \dots\dots\dots(24)$$

Hence in this case, the calculation of Eq. (22) can easily be made with the aid of the digital computer.

However if the power spectrum density of the excitation is not white, the estimation of the covariances becomes very complex procedure due to the varying of the equivalent linear parameters of the each step. Thus we shall neglect these covariances in this study for the two reasons: 1) if the peak of the power spectrum density of the excitation is not so sharp as the receptance of the structures, the covariance between the free vibration due to the previous excitation and the forced vibration due to the present excitation would be small, and 2) the free vibration dies out with time by virtue of the damping coefficient β_{eq} , so that the covariance would also become small.

To check the accuracy of the estimation of the

linear r.m.s. response derived from the step-by-step method ignoring the covariance between free and forced vibrations, the theoretical results obtained by H. Kameda¹⁾ and the results of the step-by-step method are shown in Fig. 6(b). After tedious algebraic treatments, H. Kameda obtained the theoretical r.m.s. response of a simple linear

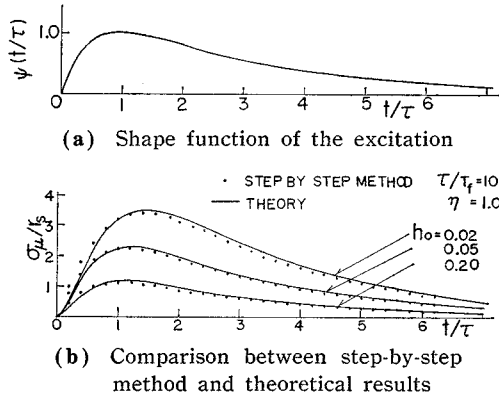


Fig. 6 Nonstationary response of linear structures.

structure subjected to an earthquake-type nonstationary excitation represented by the product of the nonstationary deterministic function shown in Fig. 6(a) and the stationary random process with the power spectrum given in Eq. (16) for the set of parameters $\eta=1.0$, $\tau/T_f=10$, $h_f=0.9$, $h_0=0.02, 0.05, 0.20$. In Fig. 6, τ denotes the equivalent duration defined by him. Both results seem to coincide well with each other except at the first and the second step.

The cause of the discrepancy between the two results at the first step is considered to be that the time derivative of the shape function shown in Fig. 6(a) is considerably great there. From this point, it is desirable that the length of the interval of the time steps is short. On the contrary, to ignore the covariance between free and forced vibrations, it is desirable that the interval is long. In this study, the section was chosen so as to furnish the same maximum r.m.s. response as that obtained by the theoretical method.

Thus the step-by-step method investigated herein seems very powerful to analyze not only nonstationary linear response but also nonlinear response of structures subjected to earthquake motion from the following merits. 1) This method can be applied to any kind of shape function. 2) The linear structure parameters can be changed at any step. 3) The process of calculation is rather easy on digital computers.

(2) Calculated Results

The r.m.s. response of structures with bilinear hysteresis subjected to nonstationary excitation has been predicted by using both the equivalent linearization technique investigated in the previous chapter and the step-by-step method in the previous section.

In this prediction, the equivalent linear parameters β_{eq} and ω_{eq} at the first time step were taken to be equal to β_0 and ω_0 ($n=0.0$), respectively, and the linear response at the first time step was obtained from Eq. (22), since the structure is considered to be in the elastic range at the start of vibration. Then β_{eq} and ω_{eq} were determined from Eq. (15) according to the response level of σ_{μ}^2 , $\sigma_{\dot{\mu}}^2$ and $\rho_{\mu\dot{\mu}}$. At the second step, the nonstationary response of the equivalent linear structure with parameters β_{eq} and ω_{eq} were calculated under the initial conditions stipulated by σ_{μ}^2 , $\sigma_{\dot{\mu}}^2$ and $\rho_{\mu\dot{\mu}}$. Then the equivalent linear parameters were determined in the same manner as that in the first step.

In Fig. 7 shown are the numerical results obtained by the above mentioned procedure for the set of parameters $\eta=1.0$, $\tau/\tau_f=3$, $h_f=0.9$, $\beta_0=$

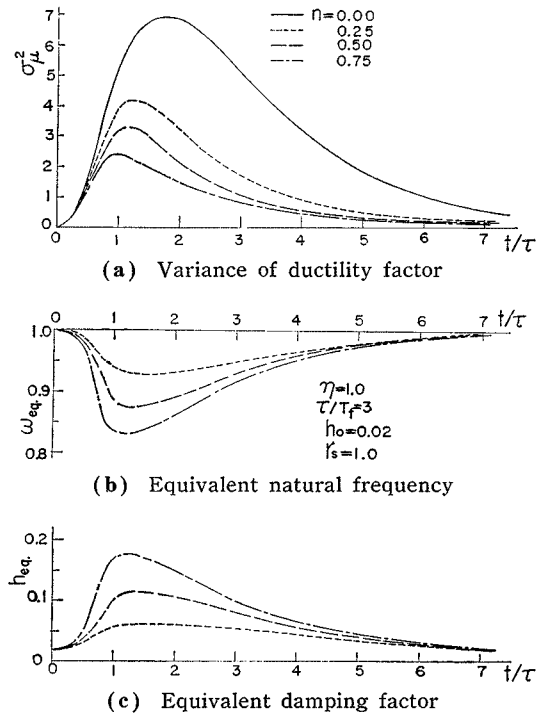


Fig. 7 Nonstationary response of structures with bilinear hysteresis.

0.04, $n=0.00, 0.25, 0.50, 0.75$. The time variation of the r.m.s. response is shown in Fig. 7 (a). It may be observed that the maximum value of the r.m.s. response of structure with bilinear hysteresis is limited and the time lag between the time of the maximum r.m.s. response and that of the maximum intensity of excitation is shortened as the nonlinearity of the hysteresis becomes strong. Especially the maximum value of the r.m.s. response of the bilinear structure with the parameter $n=0.75$ is one third of that of the linear structure and it takes place almost at the same time as the maximum intensity of excitation. It is

obvious that these results are mainly attributable to the additional damping due to the hysteresis. In other words, we could say that these results well reflect the characteristics of the energy dissipation of structures with hysteresis.

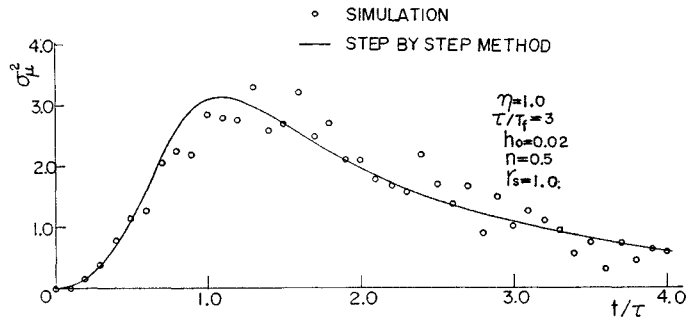
The increase in the equivalent damping factor $h_{eq} = \beta_{eq}/(2\omega_{eq})$ and the decrease in the equivalent natural frequency ω_{eq} are shown in Fig. 7 (c) and (b). As could be expected, h_{eq} grows greater and ω_{eq} smaller as the nonlinearity of the hysteresis becomes stronger.

The time lag between the maximum response and excitation becomes remarkable in general in structures with longer natural periods¹⁾. Hence the decrease in ω_{eq} takes the role of making this time lag great. However this time lag effect of ω_{eq} is so small compared with that of h_{eq} which shortens the time lag, that the maximum response of hysteretic structures occurs much earlier on the time axis than that of linear structures.

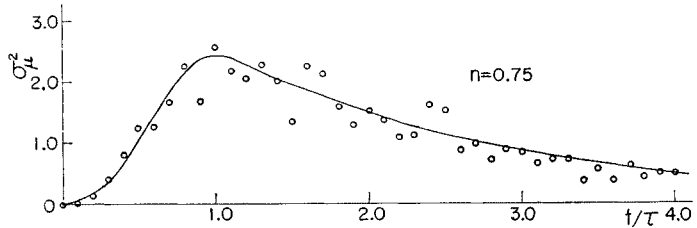
The effect of the decrease in ω_{eq} on the magnitude of the r.m.s. response would reasonably be explained by the concept of the transition of the receptance investigated in the previous chapter. In this numerical example, the natural frequency of the bilinear structure in infinitesimal vibration is the same as the predominant frequency of the excitation ω_f , so that the decrease in ω_{eq} limits the r.m.s. response.

(3) Numerical Simulation

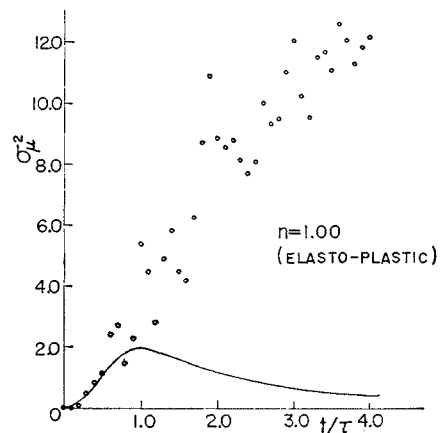
In order to check the accuracy of the analytical



(a) Bilinear hysteresis ($n=0.50$)



(b) Bilinear hysteresis ($n=0.75$)



(c) Elasto-plastic hysteresis ($n=1.00$)

Fig. 8 Comparison between simulated and predicted results of nonstationary response.

values obtained through the step-by-step linearization technique in the previous section, a numerical simulation of the nonstationary mean-square response has been carried out on a digital computer.

Earthquake-type nonstationary random excitations were generated as the product of the nonstationary deterministic shape function shown in Fig. 6 (a) and the stationary random process used in the previous chapter. The sample size for each parameter was taken as 50. The mean-

square of them at each time-step is plotted in Fig. 8. It is obvious that the analytical and the simulated results agree rather well for the parameters of $n=0.50$ and 0.75 as shown in Fig. 8(a) and (b). Especially at the beginning of the excitation, they coincide with each other fairly well, since the response of the structure is almost in the elastic region. After the response reaches the yielding point, plots of the simulated results fluctuate about the analytical values. This fluctuation is supposed to be an effect of the yielding of the structures with hysteresis. However it would be extinguished by the additional samples.

On the contrary, the discrepancy between the two results is quite clear for the structures with perfect elasto-plastic hysteresis ($n=1.0$) shown in Fig. 8(c). This discrepancy would be explained from the growth of the plastic deformation due to its excessive yielding behavior discussed in the authors' previous paper⁷⁾ in which the authors pointed out by using the moving average technique that a conspicuous plastic deformation occurs only at structures with perfect elasto-plastic hysteresis. This growth of the plastic deformation is considered as the progressing structural failure. Therefore an analytical method to predict the amount of the plastic deformation should be investigated in the near future to discuss the process of the structural collapse during strong earthquakes.

5. CONCLUSIONS

This study has dealt with the earthquake response of structures with hysteretic restoring forces through analytical and experimental methods, from which the following conclusions have been derived.

- (1) Two equivalent linearization methods have been proposed to find the general properties of earthquake response of nonlinear hysteretic structures; one is the least mean-square error method and the other is what we may call the energy balance method.
- (2) From the analysis using the slowly varying parameter method, it has been found that the final expressions of the equivalent damping coefficient and of the equivalent natural frequency obtained from these two methods have the same form.
- (3) By using the peak amplitude distribution, the stationary r.m.s. response to a Gaussian random excitation with the power spectrum density $S_f(\omega)$ has been predicted by an iterative method for structures with bilinear

hysteresis.

- (4) From the investigation of the equivalent linear parameters of hysteretic structures, it has been found that the characteristics of the r.m.s. response would reasonably be explained from the concept of the transition of the receptance due to hysteretic properties of the structures.
- (5) The step-by-step linearization technique which gives the equivalent linear parameters varying with the response level of the equivalent linear structure of the previous step has been applied to predict the nonstationary response subjected to an earthquake-type excitation.
- (6) From the error survey made with the aid of a numerical simulation on a digital computer, it can be said that the equivalent linearization techniques dealt with herein are applicable to the prediction of earthquake response of structures with bilinear hysteresis within the parameter $0.00 \leq n \leq 0.75$ and $0.0 \leq r_s \leq 1.0$.

As pointed out in this paper, the equivalent linearization techniques dealt with herein serve as powerful methods in the range of admissible error both to predict approximately the earthquake response of structures with hysteresis and to make theoretical discussions about the response properties of these structures in virtue of the hopeless difficulties in theoretical treatments of such a problem and of the time-consuming step-by-step integration of the earthquake of motion.

However, these methods cannot be applied to the structure with perfect elasto-plastic hysteresis due to the growth of the plastic deformation which could be considered as the process of the structural failure in strong earthquakes. From this viewpoint the authors are planning to investigate the amount of the plastic deformation through both theoretical and experimental methods.

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The numerical computation in this study was made on the digital computer FACOM 230-60 of the Data Processing Center, Kyoto University.

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