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STATISTICAL ESTIMATION OF THE MAXIMUM RESPONSE OF STRUCTURES SUBJECTED TO EARTHQUAKE MOTION

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ABSTRACT

In this paper, the authors derived analytically the formula finding the absolute maximum response of structures due to earthquake motions from their respective maximum acceleration. The computation results for a given structure by this formula accompanied by the assumptions of input power spectral density gave a good coincidence within a narrow band estimation with its precise response obtained by the direct-integration of the governing equation with strong-earthquake records as an input.

I. INTRODUCTION

Earthquake-resistant design of civil engineering structures with a long period of vibration has been based on their response analysis using certain strong-earthquake records or their average response spectrum. However, these methods still have some shortcomings: in the former, due to the individual earthquake characteristics, and in the latter, due to the averaging process. As a new approach, a stochastic process, characterized by spectral or correlation method of analysis, has been applied to this field of study by many researchers[1)~11)]. As a result, the expected mean value and moments of random response have been obtained. However, it is the maximum value that has the most vital role in structural design, and thus an attempt is made herein to derive the formula finding the maximum deformation response of structures when subjected to earthquake motion.

There are two methods of approach available to find the maximum value of random variables after their statistical processing: namely

(1) threshold-crossing method
(2) peak-distribution method

The general concept on (1) and (2) was formulated by S. O. Rice[12)] and D. Middleton[13)]. Furthermore S. O. Rice showed that these two methods yielded almost the same result near the maximum in the case of stationary Gaussian white noise passed by a low-pass filter or a band-pass filter. For this reason, the computation presented herein is performed by using method (2) after the spectral analysis of random variables.

This method assumes the input power spectral density of earthquake motion.

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Tajimi, for the first time, proposed to use as a substitute for this the absolute acceleration response of a one-degree-of-freedom system to a stationary Gaussian white noise[1]. But his proposal has a discrepancy in the lower frequency range near zero which severely affects the response of structures with a long vibration period. Therefore, its improved form was adopted by H. Sato[14]. In this paper the authors, taking into account the multi-predominant frequencies of earthquake motion, simulate the above spectral density of the velocity response one of an idealized foundation layer with a parallel arrangement of one-degree-of-freedom systems as an input of a stationary Gaussian white noise.

Since the intensity of earthquake motions is generally measured by their respective maximum acceleration, it is most significant to predict the corresponding maximum response of structures from this value. Tajimi, assuming that the ratios of maximum to standard deviation of both the excitation and response are equal, constructed the response amplification factors of deformation, velocity and acceleration for a structure whose natural period is shorter than twice the supposed vibration period of the foundation layer[1]. In a third case though, H. Sato corrected this by considering the extreme probability density both for the excitation and response[14]. However, in a case of structures with a long vibration period, it is the deformation response that determines their design parameters. Y. Nakao and N. Sasaki, considering the joint probability density of the above ratios, deduced a formula concerning the maximum deformation response of structures to the maximum input acceleration[15], but it still has a broad band estimation. Herein, the authors took the following approach for this investigation. The relationships between maximum and standard deviation, both of earthquake motions and response of one-degree-of-freedom systems, were first studied, and then, by combining these with the results from their spectral analyses, their precise response amplification was determined with a narrow band estimation. It is the function of (1) the response amplification in a stochastic sense of a concerned system, (2) its vibration period and the damping effect, and (3) the duration of the earthquake motion. This discovery makes it feasible to estimate properly the response of structures due to earthquake motions even in the case of a multi-degree-of-freedom system, as shown in the example.

II. STATISTICAL ESTIMATION OF THE MAXIMUM RANDOM VARIABLE

In this paper, as was mentioned in the Introduction, the absolute maximum value of random variables is found from the peak distribution. The related basic theory was reviewed by Y. K. Lin[16] in the case of a stationary stochastic process. The authors have tried further to extend it in a form applicable to a nonstationary stochastic process.

D. Middleton expressed by the following equation the expected number of peaks above the specified level during the interval of (t_1, t_2), provided that the random variable x is differentiable with respect to time up to the second order[13].

$$N(\xi, t_1, t_2)=\int_{t_1}^{t_2} |\ddot{x}(t)|\delta[\dot{x}(t)]l[x(t)-\xi]dt \tag{2.1}$$

where $\delta[\cdot]$ signifies the Dirac delta function and $l[\cdot]$ the Heaviside step function, respectively. By the use of the joint probability density function of the involved variables in Eq. (2.1), the expected number of peaks above ξ during the interval $(0, t_d)$ is obtained as

$$E[N(\xi, t_d)] = -\int_0^{t_d} dt \int_\xi^\infty \int_{-\infty}^0 \ddot{x}[p(x, \dot{x}, \ddot{x}; t)]_{\dot{x}=0} d\ddot{x}\, dx \tag{2.2}$$

where $p(x, \dot{x}, \ddot{x}; t)$, the probability density of $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ at time t, is assumed to be subjected to the Gaussian distribution and then the corresponding at $\dot{x}=0$ is expressed in the form as

$$[p(x, \dot{x}, \ddot{x}; t]_{\dot{x}=0} = \frac{1}{\sqrt{8\pi^3|M|}} \exp\left[-\frac{M_{xx}x^2 + M_{\ddot{x}\ddot{x}}\ddot{x}^2 + 2M_{x\ddot{x}}x\ddot{x}}{2|M|}\right] \tag{2.3}$$

where

$$|M| = \begin{vmatrix} \sigma_x^2 & \sigma_{x\dot{x}}^2 & \sigma_{x\ddot{x}}^2 \\ & \sigma_{\dot{x}}^2 & \sigma_{\dot{x}\ddot{x}}^2 \\ \text{symmetric} & & \sigma_{\ddot{x}}^2 \end{vmatrix} \qquad \begin{aligned} M_{xx} &= \sigma_{\dot{x}}^2 \sigma_{\ddot{x}}^2 - \sigma_{\dot{x}\ddot{x}}^4 \\ M_{\ddot{x}\ddot{x}} &= \sigma_x^2 \sigma_{\dot{x}}^2 - \sigma_{x\dot{x}}^4 \\ M_{x\ddot{x}} &= \sigma_{x\dot{x}}^2 \sigma_{\dot{x}\ddot{x}}^2 - \sigma_{\dot{x}}^2 \sigma_{x\ddot{x}}^2 \end{aligned}$$

The notations σ_x^2, $\sigma_{\dot{x}}^2$, $\sigma_{\ddot{x}}^2$; $\sigma_{x\dot{x}}^2$, $\sigma_{x\ddot{x}}^2$, $\sigma_{\dot{x}\ddot{x}}^2$ are the variances of random variables $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$, and their covariances, respectively, which are obtained through stochastic processing. The number of peaks per unit of time is found upon substitution of Eq. (2.3) into Eq. (2.2), and after some algebraic operations.

$$E[n(\xi, t)] = \frac{1}{4\sqrt{2}\pi M_{\ddot{x}\ddot{x}}} \int_\xi^\infty \left[\frac{2}{\sqrt{\pi}}\sqrt{M} \exp\left(-\frac{M_{xx}x^2}{2|M|}\right) + \frac{\sqrt{2}M_{x\ddot{x}}}{\sqrt{M_{\ddot{x}\ddot{x}}}} x \left\{1 + \text{erf}\left(\frac{M_{x\ddot{x}}x}{2|M|M_{\ddot{x}\ddot{x}}}\right)\right\} \exp\left(\frac{M_{x\ddot{x}}^2 - M_{xx}M_{\ddot{x}\ddot{x}}}{2|M|M_{\ddot{x}\ddot{x}}} x^2\right)\right] dx \tag{2.4}$$

where

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy \tag{2.5}$$

The expected total number of peaks above ξ during the interval $(0, t_d)$ is then

$$E[N(\xi, t_d)] = \int_0^{t_d} E[n(\xi, t)] dt \tag{2.6}$$

The absolute maximum value, therefore, can be found by letting the lefthand side of Eq. (2.6) equal 1.0 or 0.5, each of which gives the lower and the upper bound, respectively.

$$\text{for the lower bound:} \quad E[N(\xi_{\max}^L, t_d)] = 1.0 \tag{2.7a}$$

$$\text{for the upper bound:} \quad E[N(\xi_{\max}^U, t_d)] = 0.5 \tag{2.7b}$$

On the other hand, the probability density of peaks at time t is given, based on the heuristic assumption by Huston and Skopinski[17], as

$$p(\xi, t) = -\frac{1}{E[(n_T(t)]} \cdot \frac{\partial}{\partial \xi} E[n(\xi, t)] \tag{2.8}$$

where the notation $E[n_T(t)]$ means the expected total number of peaks per unit of time, regardless of their magnitude, and is obtained from

$$E[n_T(t)] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{0} \ddot{x}[p(x, \dot{x}, \ddot{x}; t)] d\ddot{x} \Big|_{\dot{x}=0} \tag{2.9}$$

Substitution of Eqs. (2.4) and (2.9) into Eq. (2.8) yields

$$p(\xi, t) = \frac{1}{2\sqrt{2}} \cdot \frac{M_{\ddot{x}\ddot{x}} M_{xx} - M_{x\ddot{x}}^2}{\sqrt{M_{xx} M_{\ddot{x}\ddot{x}} |M|}} \left[\frac{2}{\sqrt{\pi}} \sqrt{M} \exp\left(-\frac{M_{xx}}{2|M|} \xi^2 \right) + \frac{\sqrt{2}\, M_{x\ddot{x}}}{\sqrt{M_{\ddot{x}\ddot{x}}}} \xi \left\{ 1 + \mathrm{erf}\left(\frac{M_{x\ddot{x}}}{2|M| M_{\ddot{x}\ddot{x}}} \xi \right) \right\} \exp\left(\frac{M_{x\ddot{x}}^2 - M_{xx} M_{\ddot{x}\ddot{x}}}{2|M| M_{\ddot{x}\ddot{x}}} \xi^2 \right) \right] \tag{2.10}$$

III. SIMULATION OF EARTHQUAKE MOTION

Simulation of earthquake motion in a stochastic sense means an approximation of its spectral composition in frequency domain on the basis of strong earthquake records[18] or on the geophysical investigation. Such a procedure has been used by many researcher, assuming a filtered Guassian stationary white noise through a certain linear system[1),6),7),8)] as an earthquake motion.

The authors also followed this semi-experimental way of thinking in considering earthquake acceleration, and it is simulated by the velocity response of a linear filter[5)]

$$\ddot{x}_g + 2\mu_g \dot{x}_g + (\omega_g^2 + \mu_g^2) x_g = \ddot{x}_b \tag{3.1}$$

where the notations ω_g and μ_g represent, respectively, the predominant frequency and the damping effect of the foundation concerned. Then, applying the input-output relationship in stochastic process to the above system, the power spectral density of the synthetic earthquake motion is

$$S_{\dot{x}_g}(\omega) = |H_{\dot{x}_g}(i\omega)|^2 D \tag{3.2}$$

$$S_{\dot{x}_g}(\omega) = \frac{\omega^2}{(\omega_g^2 + u_g^2 - \omega^2)^2 + 4u_g^2\omega^2} D \tag{3.3}$$

with the intensity D at the base rock as a parameter.

Most of earthquake acceleration records, as are shown by the dotted line in Fig. 1, have a broad-band power spectral density of one peak, or that of several peaks where a few predominant frequencies can be observed. Hence, the foundation layer of an idealized parallel system of one-degree-of-freedom systems as in Fig. 2, was assumed. This results in the following input-output relationship in frequency domain.

$$S_{\dot{x}_g}(\omega) = |\sum_{i=1}^{m} \alpha_i H_{\dot{x}_{g_i}}(i\omega)|^2 D \tag{3.4}$$

where the coefficients α_i determines the magnitude among peaks. In this paper, the following simplified expression of Eq. (3.4) was adopted.

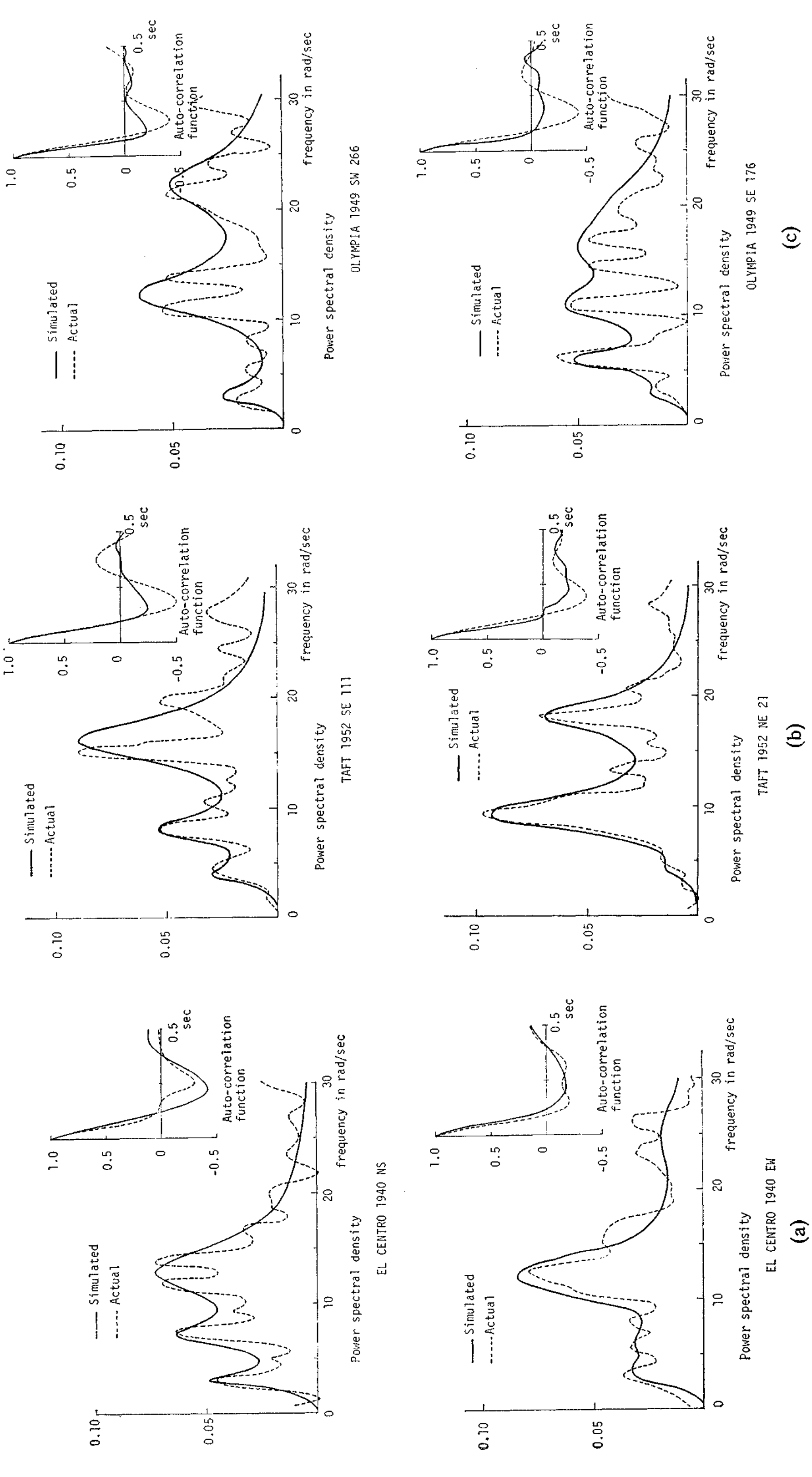

Fig. 1 Normalized earthquake characteristics

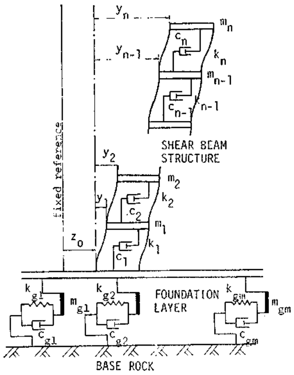

Fig. 2 Structural system

$$S_{\dot{x}_g}(\omega)=\sum_{i=1}^{m}\alpha_i\,|\,H_{\dot{x}_{g_i}}(i\omega)\,|^2 D \tag{3.5}$$

This function presents a tendency closely approximate to the results of several strong earthquake records: i.e., by adjusting parameters involved, the prominent peak or peaks of the above function can be confined within the frequency range, for instance, 0~60 rad/sec, and its amplitude can be made negligibly small near the zero frequency range, and diminished in an asymptotic manner at the higher range more than 60 rad/sec. As fundamental information concerning the value of parameters, Figs. 3, 4 are shown, where the number of superpositions was taken up to 2. Several illustrations are presented by the solid line in Fig. 1. Their conformity in shape to actual power spectral densities could be attained by choosing parameters in Eq. (3.5) as in Table 1 through man-machine communication (by FACOM 230-10).

As for the auto-correlation functions of the above synthetic earthquake motions, they are found as the Fourier transform of Eq. (3.5) as

$$R_{\dot{x}_g}(\tau)=\sum_{i=1}^{m}\frac{\alpha_i}{4\mu_{g_i}\omega_{g_i}}\exp\left(-\mu_{g_i}|\tau|\right)(\omega_{g_i}\cos\omega_{g_i}\tau-\mu_{g_i}\sin\omega_{g_i}|\tau|)D \tag{3.6}$$

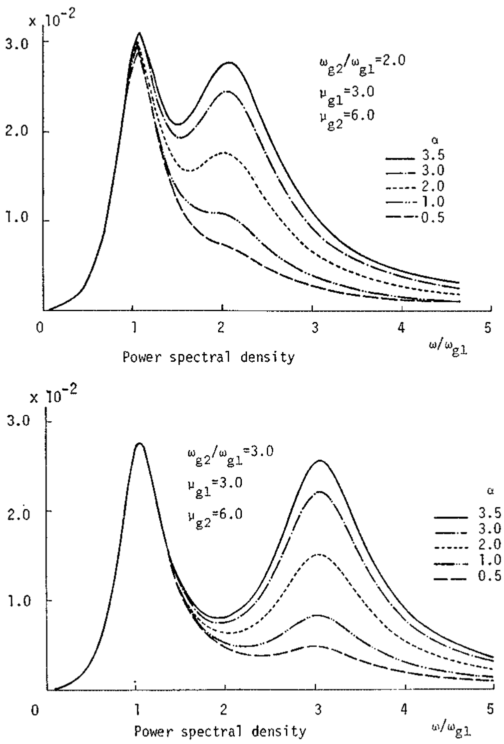

Fig. 3 Earthquake simulation effect of α

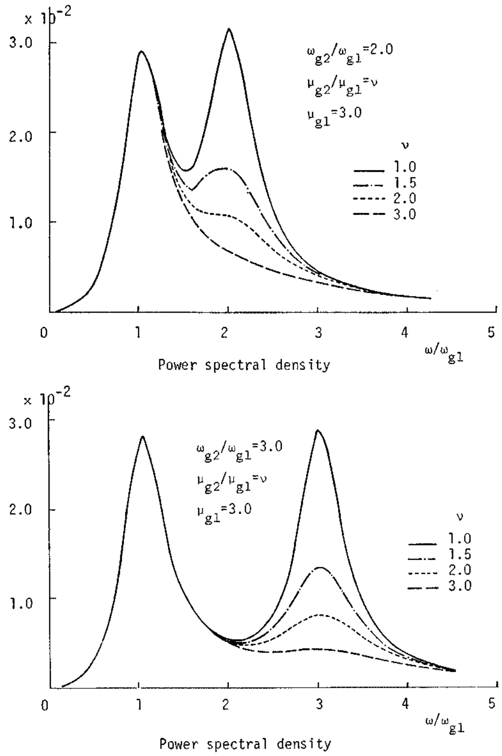

Fig. 4 Earthquake simulation effect of μ

Comparison between these auto-correlation functions and those of strong earthquake records is also made in Fig. 1.

IV. MAXIMUM ESTIMATION OF EARTHQUAKE ACCELERATION

When the earthquake motions are simulated by the stochastic process, their expected mean value and moments can easily be found. However, their maxima are more significant from the aseismic structural engineering point of view. For this investigation, the theory derived in section II is certainly applicable. The necessary statistical quantities such as those included in Eq. (2.4) or (2.10) can be obtained from Eq. (3.5) and the stationarity of the concerned process as follows:

Table 1 Value of constants used in earthquake simulation

EL CENTRO 1940 NS			EL CENTRO 1940 EW		
ω_g	μ_g	α	ω_g	μ_g	α
12.5	3.0	10000	25.0	5.0	10000
7.0	1.5	2000	12.0	2.5	13000
3.0	0.8	500	6.0	2.0	2000
			3.0	1.5	2000
TAFT 1952 SE 111			TAFT 1952 NE 21		
ω_g	μ_g	α	ω_g	μ_g	α
16.0	2.5	10000	18.0	2.0	8000
8.0	1.5	2000	9.0	2.0	12000
4.0	1.0	500	4.0	1.0	300
OLYMPIA 1949 SE176			OLYMPIA 1949 SW266		
ω_g	μ_g	α	ω_g	μ_g	α
20.0	3.5	1500	22.0	2.5	7000
16.0	3.0	2000	12.0	2.5	9000
11.0	2.0	1200	3.0	1.5	1000
6.0	1.0	300			
3.0	1.0	100			

$$\sigma_F^2 = \frac{1}{2\pi} \sum_{i=1}^{m} \int_0^\infty |H_{\dot{x}_{g_i}}(i\omega)|^2 D \, d\omega \qquad \text{in the case of one-sided power spectral density[19]} \tag{4.1.a}$$

$$\sigma_F^2 = \frac{1}{2\pi} \sum_{i=1}^{m} \int_{-\infty}^\infty |H_{\dot{x}_{g_i}}(i\omega)|^2 D \, d\omega \qquad \text{in the case of two-sided power spectral density} \tag{4.1.b}$$

$$\sigma_{\dot{F}}^2 = \frac{1}{2\mu} \sum_{i=1}^{m} \int_0^\infty \omega^2 |H_{\dot{x}_{g_i}}(i\omega)|^2 D \, d\omega \tag{4.2}$$

$$\sigma_{\ddot{F}}^2 = \frac{1}{2\pi} \sum_{i=1}^{m} \int_0^\infty \omega^4 |H_{\dot{x}_{g_i}}(i\omega)|^2 D \, d\omega \tag{4.3}$$

$$\sigma_{F\dot{F}}^2 = 0 \tag{4.4}$$

$$\sigma_{\dot{F}\ddot{F}}^2 = 0 \tag{4.5}$$

$$\sigma_{F\ddot{F}}^2 = -\sigma_{\dot{F}}^2 \tag{4.6}$$

The actual calculations were carried out with an integral range of 0~60 rad/sec. This truncation is efficacious as is proved by comparing the results by numerical integration of Eq. (4.1.a) and by residue integration of Eq. (4.1.b).

Substitution of Eq. (4.1) through Eq. (4.6) into Eq. (2.4) and Eq. (2.10) yields the expected number of peaks of the synthetic earthquake motion above the specified level ξ and its probability density of peaks, which are shown in Figs. 5 and 6, respectively. Fig. 5 indicates that the maximum acceleration of earthquake motion is about 3.0~3.5 times of its standard deviation, regardless of the shape of its power spectral density, but affected by its duration as shown in Fig. 7. Furthermore, this figure predicts the following approximate relationship between the ratio of maximum to standard deviation of earthquake acceleration and its duration.

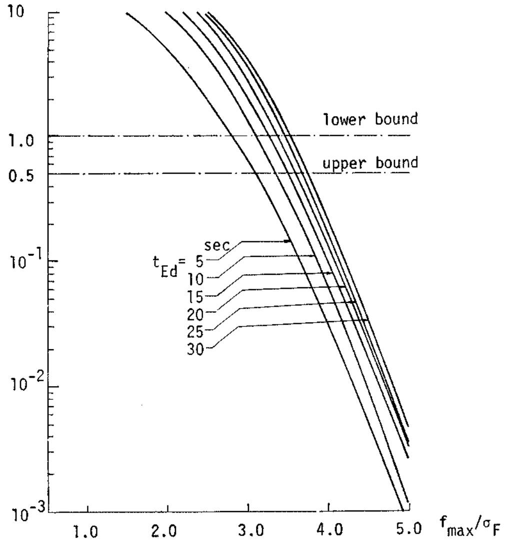

Fig. 5 Peak distribution of earthquake motion

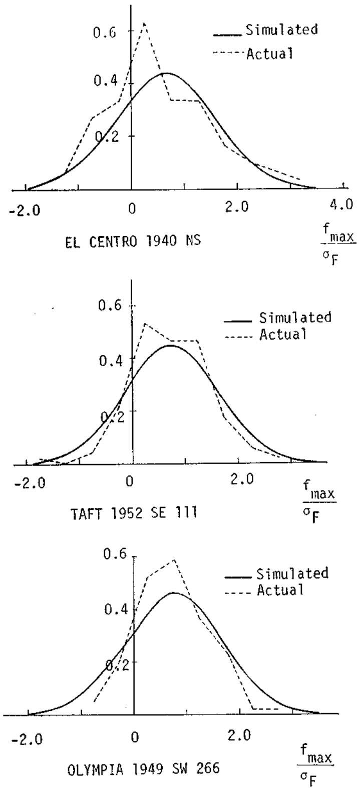

Fig. 6 Probability density of peaks

for the lower bound

$$\alpha_f^L = 0.91 \log_{10} t_{Ed} + 2.16 \qquad (4.7.a)$$

for the upper bound

$$\alpha_f^U = 0.82 \log_{10} t_{Ed} + 2.53 \qquad (4.7.b)$$

These formulae have a narrow-band estimation of the maximum. Their application to strong earthquake records, as indicated in Table 2, gives a good coinci-

Table 2 Estimation of the absolute maximum acceleration of earthquakes

	t_{Ed}	f^{ob}_{max}	f^{ob}_{min}	σ_f	α^L_f	$f^{st}_{L,max}$	α^U_f	$f^{st}_{U,max}$	(1)	(2)	(3)	(4)
EL CENTRO 1940 NS	10.3	298	-318	94.3	3.11	293	3.38	319	1.02	1.08	0.93	1.00
EL CENTRO 1940 EW	29.0	230	-160	70.2	3.49	245	3.73	261	0.94	0.65	0.88	0.61
TAFT 1952 SE 111	17.0	180	-149	57.0	3.28	187	3.53	201	0.96	0.80	0.90	0.74
TAFT 1952 NE 21	16.0	170	-182	56.6	3.26	184	3.51	199	0.92	0.99	0.85	0.91
OLYMPIA 1949 SE 176	19.0	160	-193	69.2	3.32	230	3.58	247	0.70	0.84	0.65	0.78
OLYMPIA 1949 SW 266	19.0	307	-193	84.4	3.32	281	3.58	302	1.09	0.69	1.02	0.64

(1)= $f^{ob}_{max}/f^{st}_{L,max}$ (2)= $-f^{ob}_{min}/f^{st}_{L,max}$ (3)= $f^{ob}_{max}/f^{st}_{U,max}$ (4)= $-f^{ob}_{min}/f^{st}_{U,max}$

t_{Ed} : duration of earthquake record used for statistical analysis

f^{ob}_{max} : observed maximum acceleration

f^{ob}_{min} : observed minimum acceleration

σ_f : standard deviation of peaks of earthquake records

α^L_f : lower bound ratio of maximum to standard deviation, Eq.(4.7.a)

α^U_f : upper bound ratio of maximum to standard deviation, Eq.(4.7.b)

$f^{st}_{L,max}$: lower bound of the statistical maximum acceleration

$f^{st}_{U,max}$: upper bound of the statistical maximum acceleration

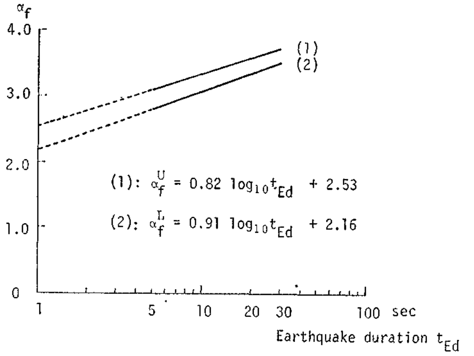

Fig. 7 Ratio of earthquake maximum acceleration to its standard deviation

dence of actually observed maxima or minima. In Fig. 6 are also presented the peak distribution directly obtained from strong earthquake records for comparison. Each assumed stochastic process well represents the significant range of peak distribution—the portion which is more than 1.5 times the standard deviation, and it is concluded that the above good estimates of the maxima or minima of earthquake accelerations are due to this fact.

V. RESPONSE OF A ONE-DEGREE-OF-FREEDOM SYSTEM

The input-output relationship of a one-degree-of-freedom system in frequency domain, when restricted within an elastic limit, is expressed by

$$S_R(\omega)=|{}^d_aH(\underline{i}\omega)|^2 S_F(\omega) \tag{5.1}$$

where S_F and S_R are the power spectral densities of excitation and response, respectively. ${}^d_aH(\omega)$ is the transfer function with regard to displacement response of one-degree-of-freedom system to acceleration input; i.e.,

$$ {}^d_aH(\underline{i}\omega)=\frac{-1}{(\omega_0^2-\omega^2)+\underline{i}2\zeta\omega_0\omega} \tag{5.2}$$

where ω_0 and ζ are the undamped natural frequency and the damping factor, respectively. When the system is subjected to the synthetic earthquake motion of Eq. (3.5), its two-sided response power spectral density is thus obtained as

$$S_R(\omega)=\sum_{i=1}^{m}\alpha_i|H(\underline{i}\omega)|^2|H_{\ddot{x}_{g_i}}(\underline{i}\omega)|^2 D/2 \tag{5.3}$$

The corresponding variance, as the inverse Fourier transform of this, is

$$\sigma_d^2(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_R(\omega)\,d\omega \tag{5.4}$$

Integration of Eq. (5.4) upon substitution of Eq. (5.3) is carried out by the method of residue theory. Such integral formulae applicable here are given in reference (20) in the following form:

$$I=\int_{-\infty}^{\infty}|H(\underline{i}\omega)|^2 d\omega \tag{5.5}$$

where

$$H(\underline{i}\omega)=\frac{-\underline{i}\omega^3B_3-\omega^2B_2+\underline{i}\omega B_1+B_0}{\omega^4A_4-\underline{i}\omega^3A_3-\omega^2A_2+\underline{i}\omega A_1+A_0} \tag{5.6}$$

and the result is

$$I=\pi[(B_0^2/A_0)(A_2A_3-A_1A_4)+A_3(B_1^2-2B_0B_2)+A_1(B_2^2-2B_1B_3)$$
$$+(B_3^2/A_4)(A_1A_2-A_0A_3)]/[A_1(A_2A_3-A_1A_4)-A_0A_3^2] \tag{5.7}$$

This kind of residue integrals were originally formulated by H. M. James and others[21]. Then, the displacement response variance is

$$\sigma_d^2=\frac{1}{2\pi}\sum_{i=1}^{m}\frac{I_i}{2} \tag{5.8}$$

where I_i is the integrated value by the use of Eq. (5.7) with the following coefficients:

$$\left.\begin{aligned} &A_{0i}=\omega_0^2(\omega_{g_i}^2+\mu_{g_i}^2) && B_{0i}=0 \\ &A_{1i}=2\omega_0\{\zeta(\omega_{g_i}^2+\mu_{g_i}^2)+\mu_{g_i}\omega_0\} && B_{1i}=-1 \\ &A_{2i}=\omega_0^2+\omega_{g_i}^2+\mu_{g_i}^2+4\zeta\omega_0\mu_{g_i} && B_{2i}=0 \\ &A_{3i}=2(\zeta\omega_0+\mu_{g_i}) && B_{3i}=0 \\ &A_{4i}=1 && \end{aligned}\right\} \tag{5.9}$$

The velocity response variance is likewise

$$\sigma_v^2=\frac{1}{2\pi}\sum_{i=1}^{m}\int_{-\infty}^{\infty}\omega^2|{}_a^dH(i\omega)|^2\alpha_i|H_{\dot{x}_{g_i}}(i\omega)|^2D/2\cdot d\omega \tag{5.10}$$

This integration is also carried out by Eq. (5.7) with the same coefficients of A_{ji} $(j=1,4)$ as in Eq. (5.9) and with B_{ji} $(j=1,4)$ of

$$\left.\begin{aligned} &B_{0i}=0 \\ &B_{1i}=0 \\ &B_{2i}=-1 \\ &B_{3i}=0 \end{aligned}\right\} \tag{5.11}$$

As for the absolute acceleration response variance, using the transfer function of

$$\sigma_a^2=\frac{1}{2\pi}\sum_{i=1}^{m}\int_{-\infty}^{\infty}|{}_a^dH(i\omega)|^2\alpha_i|H_{\dot{x}_{g_i}}(i\omega)|^2D/2\cdot d\omega \tag{5.12}$$

instead of Eq. (5.2), it is found to be

$${}_a^dH(i\omega)=\frac{\omega_0^2+i2\zeta\omega_0\omega}{\omega_0^2-\omega^2+i2\zeta\omega_0\omega} \tag{5.13}$$

The integrated value is obtained by Eq. (5.7) with the same coefficients of A_{ji} $(j=1,4)$ in Eq. (5.9) and the following B_{ji}:

$$\left.\begin{aligned} &B_{0i}=0 \\ &B_{1i}=\omega_0^2 \end{aligned}\right\} \tag{5.14}$$

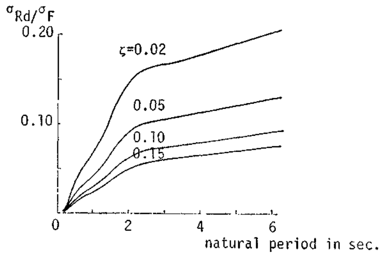

Fig. 8 Deformation amplification spectrum by stochastic analysis

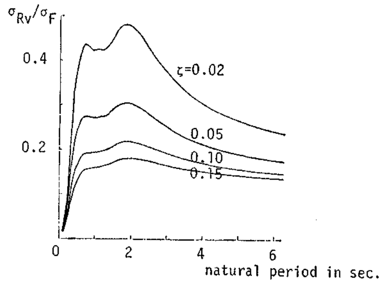

Fig. 9 Velocity amplification spectrum by stochastic analysis

$$\left.\begin{aligned} B_{2i}&=2\zeta\omega_0 \\ B_{3i}&=0 \end{aligned}\right\}$$

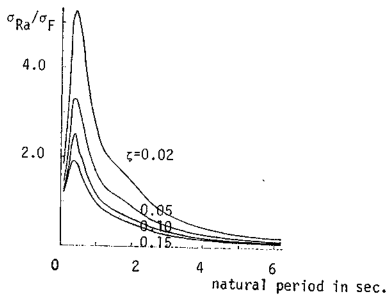

Fig. 10 Acceleration amplification spectrum by stochastic analysis

Figs. 8, 9, 10, which were obtained through averaging the response to the synthetic earthquake motions in section III, represent the response amplification factors in a statistical sense with regard to deformation, velocity and absolute acceleration, respectively. Their general features closely approximate those to actual strong earthquakes[22] in the range less than about 3 rad/sec. The precise maximum spectra, however, may be the different ones from the above figures in the range more than that, especially so the deformation spectrum as is investigated below.

VI. MAXIMUM RESPONSE OF A ONE-DEGREE-OF-FREEDOM SYSTEM

Maximum response of a lightly damped system is more easily calculated by applying the Rayleigh distribution rather than by Eq. (2.10) for its peak distribution of response. The former distribution is a special case of the latter with a random index $\varepsilon=0$ [23]. In this case the number of peaks is supposedly between the number of zero-crossing with upward (or downward) and its twice. Hence, the following equations with regard to the lower and the upper bounds of the number of peaks above ξ.

for the lower bound

$$E[N^L(\xi, t_d)]=\int_0^{t_{\mathrm{Rd}}} E[n(0+)]\int_\xi^\infty p(x)dx\,dt \tag{6.1.a}$$

for the upper bound

$$E[N^U(\xi, t_d)]=\int_0^{t_{\mathrm{Rd}}} 2E[n(0+)]\int_\xi^\infty p(x)dx\,dt \tag{6.1.b}$$

where the number of zero-crossing with upward is obtained by Eq. (2.9) as

$$E[n(0+)]=\frac{1}{2\pi}\cdot\frac{\sigma_v}{\sigma_d} \tag{6.2}$$

and the Rayleigh distribution function is expressed by

$$p(x)=\frac{x}{\sigma_d^2}\exp\left(-\frac{x^2}{2\sigma_d^2}\right) \tag{6.3}$$

After integration of Eq. (6.1) over the variable x from ξ to infinite, it becomes as

for the lower bound

$$E[N^L(\xi, t_d)]=\int_0^{t_{Rd}} E[n(0+)]\exp\left(-\frac{\xi^2}{2\sigma_d^2}\right)dt \tag{6.4.a}$$

for the upper bound

$$E[N^U(\xi, t_d)]=\int_0^{t_{Rd}} 2E[n(0+)]\exp\left(-\frac{\xi^2}{2\sigma_d^2}\right)dt \tag{6.4.b}$$

Maximum response during the motion can thus be found by letting the righthand side of Eq. (6.4) equal 1, which leads the following ratio of maximum to standard deviation response:

for the lower bound

$$\frac{d^L_{\max}}{\sigma_d}=\sqrt{2\log_e(E[n(0+)]t_{Rd})} \tag{6.5.a}$$

for the upper bound

$$\frac{d^U_{\max}}{\sigma_d}=\sqrt{2\log_e(2E[n(0+)]t_{Rd})} \tag{6.5.b}$$

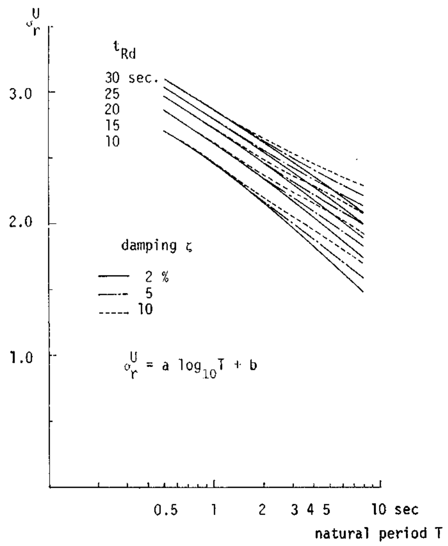

Fig. 11 Ratio of maximum response to its standard deviation (upper bound)

These ratios have almost identical tendencies regardless of the input power spectral density as are shown in Fig. 11. They very proportionally in a logarithmic as the natural period of the system. Hence, the following formulae are propose for them:

for the lower bound

$$\alpha_r^L=a^L\log_{10}T+b^L \tag{6.6.a}$$

for the upper bound

$$\alpha_r^U=a^U\log_{10}T+b^U \tag{6.6.b}$$

where the coefficients a and b, which are the function of the response duration of the system and its damping effect, are listed in Table 3.

Substitution of Eqs. (4.7) and (6.6) into the result by spectral analysis of a given system yields its maximum deformation

Table 3 Ratio of response maximum to its standard deviation

Response duration t_{Rd}= 10 sec.

damping factor ζ(%)	L.B. a^L	L.B. b^L	U.B. a^U	U.B. b^U
2	-1.20	2.15	-1.03	2.45
5	-1.10	2.16	-0.96	2.45
10	-1.00	2.15	-0.84	2.45
15	-0.93	2.17	-0.80	2.46

Response duration t_{Rd}= 15 sec.

damping factor ζ(%)	L.B. a^L	L.B. b^L	U.B. a^U	U.B. b^U
2	-1.07	2.33	-0.94	2.61
5	-1.00	2.33	-0.86	2.60
10	-0.88	2.34	-0.77	2.62
15	-0.82	2.35	-0.73	2.63

Response duration t_{Rd}= 20 sec.

damping factor ζ(%)	L.B. a^L	L.B. b^L	U.B. a^U	U.B. b^U
2	-0.95	2.45	-0.90	2.72
5	-0.91	2.44	-0.82	2.71
10	-0.83	2.46	-0.74	2.74
15	-0.77	2.48	-0.69	2.74

Response duration t_{Rd}= 25 sec.

damping factor ζ(%)	L.B. a^L	L.B. b^L	U.B. a^U	U.B. b^U
2	-0.96	2.55	-0.87	2.80
5	-0.84	2.54	-0.78	2.80
10	-0.79	2.55	-0.72	2.81
15	-0.73	2.56	-0.67	2.83

Response duration t_{Rd}= 30 sec.

damping factor ζ(%)	L.B. a^L	L.B. b^L	U.B. a^U	U.B. b^U
2	-0.94	2.61	-0.83	2.86
5	-0.85	2.61	-0.75	2.87
10	-0.78	2.63	-0.69	2.89
15	-0.71	2.63	-0.67	2.89

response as the function of input maximum acceleration.

$$d_{\max}=\left(\frac{\alpha_r}{\alpha_f}\right)\left(\frac{\sigma_R}{\sigma_F}\right)f_{\max} \qquad (6.7)$$

where (σ_R/σ_F) is the response amplification factor in Fig. 8. In Fig. 12 the deformation spectrum obtained herein is compared with the one proposed by Y. Nakao and N. Sasaki[15], and the one by the Research Institute of Construction Ministry of Japan[24], where the maximum input acceleration was set to equal 200 gal. The result in this paper shows a flat spectrum against others in the range

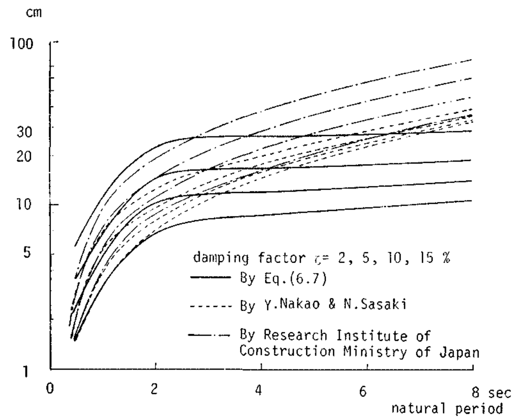

Fig. 12 Deformation spectra
Maximum acceleration of earthquake was set to eqaul 200 gal

more than 2.5 sec of the natural period. This tendency is more rational from the fact that earthquake motions appear directly in the system response when the system has a long vibration period one.

VII. MAXIMUM RESPONSE OF A MULTI-DEGREE-OF-FREEDOM SYSTEM WITH APPLICATION TO A MULTI-STORY BUILDING

Response analysis of lightly damped structure with multi-degree-of-freedom is usually carried out by the modal method, with the displacement expressed as

$$y_k(t)=\sum_{i=1}^{n} q_i(t)\phi_k{}^{(i)} \tag{7.1}$$

where n is the number of degree-of-freedom of a concerned structure and $\phi_k{}^{(i)}$ is the i-th mode shape function of its k-section. The time function $q_i(t)$ is governed by the following equation.

$$\ddot{q}_i+2\zeta_i\omega_{0i}\dot{q}+\omega_{0i}^2 q_i=-\beta_i f \tag{7.2}$$

with notations ω_{0i} and ζ_i being the corresponding undamped natural frequency and damping factor, respectively. The participation factor β_i shares the i-th modal contribution to the structural response and it is obtained as

$$\beta_i=\frac{\sum_{k=1}^{n} m_k\phi_k{}^{(i)}}{\sum_{k=1}^{n} m_k\{\phi_k{}^{(i)}\}^2} \tag{7.3}$$

where m_k is the concentrated mass at k-section of the structure. However, in the case of random excitation input such as earthquake motions, the above superposition of Eq. (7.1) must be replaced by the following through spectral analysis.

$$\sigma_{y_k}^2(t)=E[y_k{}^2(t)] \tag{7.4}$$

$$\sigma_{y_k}^2(t)=[\sum_{i=1}^{n} J_{ii}\{\beta_i\phi_k{}^{(i)}\}^2+2\sum_{i<j}^{n} J_{ij}\phi_k{}^{(i)}\phi_k{}^{(j)}]\sigma_F{}^2 \tag{7.5}$$

where

$$J_{ii}=\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}|{}_a^dH_i(i\omega)|^2 S_F(\omega)d\omega\right]\Big/\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}S_F(\omega)\,d\omega\right] \tag{7.6}$$

$$J_{ij}=\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{R}({}_a^dH_i(i\omega){}_a^dH_i{}^*(i\omega))S_F(\omega)d\omega\right]\Big/\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}S_F(\omega)\,d\omega\right] \tag{7.7}$$

$$\sigma_F{}^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_F(\omega)\,d\omega \tag{7.8}$$

$${}_a^dH(i\omega)=\frac{-1}{\omega_{0i}^2-\omega^2+i2\zeta_i\omega_{0i}\omega} \tag{7.9}$$

In Eq. (7.5), the first part is the direct-contributions of each normal mode, and the second, the cross-contributions between them. The latter becomes significantly large in proportion to the closeness between adjacents modes and the value of their damping effects: i.e.[8],

$$\text{If} \quad 1-\left(\frac{\omega_i}{\omega_j}\right)^2 \gg 4\beta_j^2\left\{\left(\frac{\omega_i}{\omega_j}\right)+2\right\}; \qquad \omega_j>\omega_i, \quad \beta_j=\beta_i \tag{7.10}$$

then the cross-contribution between i-th and j-th mode is negligibly small

Moreover, according to the results from investigation of a tower and pier system of a long span suspension bridge[8], structures of good performance from the aseismic design point of view are the ones with their vibration modes sufficiently separated. Then, adopting the same consideration in the previous section, the maximum response for this case is obtained as

$$y_{\kappa\,\max}=\sqrt{\sum_{i=1}^{n} \alpha_{r_i}^2 \gamma_i^2 \{\beta_i \phi_\kappa^{(i)}\}^2} \cdot \sigma_F \tag{7.11}$$

where the notation α_{r_i} coincides with α_r in Eq. (6.7) at the natural period of the i-th mode of the concerned structure, and γ_i is the deformation amplification factor in a stochastic sense with regard to i-th mode. This formula is expressed as the function of the maximum input acceleration upon substitution of Eq. (4.7).

$$y_{\kappa\,\max}=\sqrt{\sum_{i=1}^{n} \alpha_{r_i}^2 \gamma_i^2 \{\beta_i \phi_\kappa^{(i)}\}^2} \cdot f_{\max}/\alpha_f \tag{7.12}$$

Example

A 33-story building, idealized with an equivalent 5-mass system was analyzed as an illustrative example of the above theory. Its dynamic properties are presented in Fig. 13 and Table 4. The design details are contained in reference (25).

In Table 5 are listed the computation results by the direct integration[26]

Table 4

vibration modes	1-st	2-nd	3-rd	4-th	5-th
natural period (sec.)	4.20	1.73	1.14	0.87	0.70

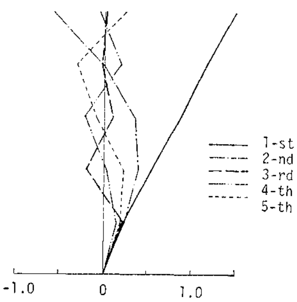

Fig. 13 Mode shapes

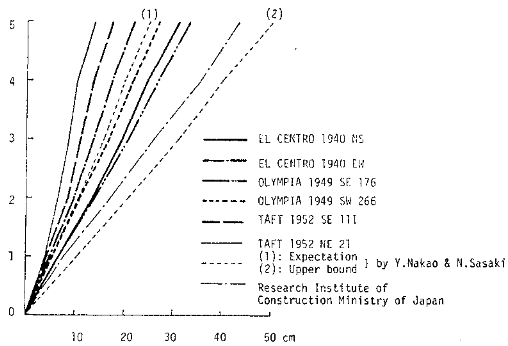

Fig. 14 Maximum deformation response
Maximum input acceleration 200 gal
Response duration $t_{Rd}=25$ sec
Fraction of damping $\zeta=0.05$ of each mode

Table 5 Maximum response

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	Direct-integrated response(by β-method)	Response by Eq.(7.12)* L.B.	U.B.	Response by Y.Nakao & N.Sasaki Expectation	U.B.
SEC.1	4.6 cm	5.3 cm	6.5 cm	9.1 cm	18.2 cm
SEC.2	10.1	10.8	13.3	19.2	38.3
SEC.3	17.3	15.5	19.3	28.4	56.8
SEC.4	25.0	19.8	24.6	36.6	73.2
SEC.5	30.8	25.1	31.1	45.0	91.9

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	Direct-integrated response(by β-method)	Response by Eq.(7.12)* L.B.	U.B.	Response by Y.Nakao & N.Sasaki Expectation	U.B.
SEC.1	4.1 cm	3.2 cm	3.6 cm	4.7 cm	9.4 cm
SEC.2	5.7	5.5	6.3	8.7	17.3
SEC.3	8.1	7.5	8.6	12.1	24.3
SEC.4	9.6	9.3	10.7	15.3	30.6
SEC.5	12.3	12.0	13.7	19.5	40.0

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	Direct-integration response(by β-method)	Response by Eq.(7.12)* L.B.	U.B.	Response by Y.Nakao & N.Sasaki Expectation	U.B.
SEC.1	4.7 cm	3.8 cm	4.3 cm	5.7 cm	11.4 cm
SEC.2	8.6	7.1	8.1	11.2	22.5
SEC.3	11.5	9.5	11.0	15.5	31.1
SEC.4	13.3	11.5	13.3	19.3	38.6
SEC.5	16.0	15.2	17.5	25.1	50.1

* Response duration 25 sec.

of the governing equation of the concerned structure with an input excitation of actual strong earthquake records, and the results by using Eq. (7.12) and by the proposal of Y. Nakao[15] with an input of their respective simulated power spectral densities. Our results show a good coincidence with a narrow band estimation to the precise structural response to earthquakes. This fact guarantees Eq. (7.12) strongly for estimation of the maximum response of structures to earthquake motions. Fig. 14 presents the comparison among maximum response by our proposing formula, by Y. Nakao[15], and by the Research Institute of Construction Ministry of Japan[24], indicating that structural response is severely affected by the input power spectral density of earthquake motion. Therefore, the precise determination of the former requires the latter of well estimated. This will be possible through analytical or experimental geophysical investigations of the construction site of a structure.

VIII. CONCLUSIONS

The following conclusions are derived based on the results of this analytical study.

1. Simulation of earthquake motion adopted in this paper proved to be valid and useful in structural response analysis. It makes possible any arbitrary shaped function for the power spectral density of earthquake motion.

2. The ratio of maximum acceleration of earthquake motion, f_{max} to its

standard deviation, σ_f, regardless of the shape of its power spectral density, can be expressed only as a function of its duration t_{Ea}, as

for the lower bound $\sigma_f{}^L = 0.91 \log_{10} t_{Ea} + 2.16$ (8.1.a)

for the upper bound $\sigma_f{}^U = 0.82 \log_{10} t_{Ea} + 2.53$ (8.1.b)

3. The ratio of maximum response of a one-degree-of-freedom system, R_{max} to its standard deviation response, σ_R can be expressed as a function of its vibration period, T, as

$$\alpha_r = a \log_{10} T + b \tag{8.2}$$

where the coefficients a and b are different according to the response duration of the system and its damping effect, and are listed in Table 3.

4. Combination of the above results (1) and (2) with the input-output relationship in frequency domain of a concerned system provides the following formula as to its maximum response.

$$d_{max} = \left(\frac{\alpha_r}{\alpha_f}\right)\left(\frac{\alpha_R}{\alpha_F}\right) f_{max} \tag{8.3}$$

This formula can readily be extended to a structure with multi-degree-of-freedom system when it has well separated vibration modes.

$$y_{max} = \sqrt{\sum_i^n \alpha_{r_i}^2 \gamma_i \{\beta_i \phi^{(i)}\}^2} \cdot f_{max}/\alpha_f \tag{8.4}$$

5. Computation results of a 33-story building gave a good coincidence to the response obtained by direct integration of its governing equation as an input of strong earthquake records.

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