PROBABILITY OF STRUCTURAL FAILURE UNDER EARTHQUAKE ACCELERATION

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SYNOPSIS

A method is presented to find upper and lower bounds of the probability that a simple structure will fail under a random loading of earthquake type. The structure considered is a one-story building treated as a system of mass, spring and dashpot with single-degree of freedom. The failure is assumed to occur when the absolute value of the displacement of the mass exceeds a critical value. The present method leads to evaluation of lengthy integrals which, however, can be performed numerically with an electronic digital computer and produces the upper and lower bound close enough at least for the order estimation of the probability.

1. INTRODUCTION

When failure of a structure initially at rest and subject to a random excitation is considered, the probability $P_x(T;\ -\lambda_1,\ \lambda_2)$ that the response $x(t)$ of the structure such as stress, strain or deflection will exceed specified negative and positive threshold values $-\lambda_1(<0)$ and $\lambda_2(>0)$ in the time interval $(0,\ T)$ is, as pointed out by Bogdanoff, Goldberg and Bernard[1] and by Bolotin[2], of primary importance. For example, if failure of a structure occurs when the stress σ at critical section exceeds the "ultimate strength" either in compression $-\sigma_b$ or in tension σ_t, then the probability of failure of the structure in the interval $(0, T)$ is given by $P_\sigma(T;\ -\sigma_b,\ \sigma_t)$.

This is essentially the problem of finding the distribution of the first passage time and exact solution seems extremely difficult to obtain.

However, it is possible to evaluate an upper and lower bound of $P_x(T;\ -\lambda_1,\ \lambda_2)$, the derivation of which has been given in Reference 3 and is repeated here briefly.

If $\{A\} \subset \{B\}$ represents the statement that the event A belongs to the event B, it is evident that

$$\{x(\tau_1)<-\lambda_1\} \subset \{\min x(t)<-\lambda_1\}$$

and

$$\{x(\tau_2)>\lambda_2\} \subset \{\max x(t)>\lambda_2\}$$

for any time τ_1 and τ_2 $(0\leqq\tau_1,\ \tau_2\leqq T)$ where $0\leqq t\leqq T$.

Therefore, if $\{A\cup B\}$ denotes the union of the events A and B,

$$\begin{aligned} &P_x(T;\ -\lambda_1,\ \lambda_2) \\ &\quad = P\left[\{\min x(t)<-\lambda_1\} \cup \{\max x(t)>\lambda_2\}\right] \\ &\quad\ > P\left[\{x(\tau_1)<-\lambda_1\} \cup \{x(\tau_2)>\lambda_2\}\right] \end{aligned} \quad \cdots\cdots(1)$$

where $0\leqq\tau_1,\ \tau_2\leqq T$ and P represents the probability of the event indicated in the braces or brackets following it. Hence, the last member of Eq. (1) is a lower bound for $P_x(T;\ -\lambda_1,\lambda_2)$.

To find an upper bound, $P_x(T;\ -\infty,\ \lambda)$ $(\lambda>0)$ is considered first. By definition,

$$P_x(T;\ -\infty,\ \lambda) = P\{\max x(t)\geqq\lambda\} \quad (t\leqq T) \quad \cdots\cdots(2)$$

If $\{A, B\}$ denotes the joint event of A and B,

$$\begin{aligned} &P_x(T+dT;\ -\infty,\ \lambda) \\ &\quad = P_x(T;\ -\infty,\ \lambda) \\ &\qquad + P\{\max x(t)<\lambda,\ \max x(t')\geqq\lambda\} \end{aligned} \quad \cdots\cdots(3)$$

and

$$\begin{aligned} &P\{\max x(t)<\lambda,\ \max x(t')\geqq\lambda\} \\ &\quad < P\{x(T)<\lambda,\ \max x(t')\geqq\lambda\} \end{aligned} \quad \cdots\cdots(4)$$

where t and t' are such that $0<t\leqq T$ and $T<t'\leqq T+dT$.

The last expression represents the probability that $x(t)$ crosses λ with positive slope in the interval $(T,\ T+dT)$. Hence, following Rice[4]

$$\begin{aligned} &P\{x(T)<\lambda,\ \max x(t')\geqq\lambda\} \\ &\quad = \int_0^\infty \dot{x}\,\phi(\lambda,\dot{x};\ T)\,d\dot{x}\,dT \end{aligned} \quad \cdots\cdots(5)$$

where $T<t'\leqq T+dT$ and $\phi(x,\dot{x};\ T)$ is the

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joint density function of $x=x(T)$ and $\dot{x}=\dot{x}(T)=dx(T)/dT$.

From Eqs. (3) ~ (5),

$$\frac{dP_x(T;\ -\infty,\ \lambda)}{dT} < \int_0^\infty \dot{x}\,\phi(\lambda,\ \dot{x};\ T)\,d\dot{x} \equiv h_2(T) \quad \cdots(6)$$

Similarly,

$$\frac{dP_x(T;\ -\lambda,\ \infty)}{dT} < \int_{-\infty}^0 |\dot{x}|\phi(-\lambda,\ \dot{x};\ T)\,d\dot{x} \equiv h_1(T) \quad \cdots\cdots(7)$$

Hence, for $0 \leq t \leq T$,

$$\begin{aligned} P_x(T;\ -\lambda_1,\ \lambda_2) &= P\left[\{\min x(t) < -\lambda_1\} \cup \{\max x(t) > \lambda_2\}\right] \\ &< P_x(T;\ -\lambda_1,\ \infty) + P_x(T;\ -\infty,\ \lambda_2) \\ &< \int_0^T [h_1(\tau)+h_2(\tau)]\,d\tau \quad \cdots\cdots(8) \end{aligned}$$

The last expression is an upper bound for $P_x(T;\ -\lambda_1,\ \lambda_2)$.

The use of Eqs. (1) and (8) for the bounds does not require $x(t)$ to be Gaussian, or of white noise type or stationary.

However, as will be discussed in the following sections, the response of the structure to the excitation is assumed to be Gaussian with the joint density function

$$\begin{aligned} \phi(x,\dot{x};\ T) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\right. \\ &\quad \left.\times\left\{\left(\frac{x}{\sigma_1}\right)^2 - \frac{2\rho x\dot{x}}{\sigma_1\sigma_2} + \left(\frac{\dot{x}}{\sigma_2}\right)^2\right\}\right] \quad \cdots(9) \end{aligned}$$

where $\sigma_1=\sigma_1(T)$, $\sigma_2=\sigma_2(T)$ are the standard deviation and $\rho=\rho(T)$ the correlation function of $x(T)$ and $\dot{x}(T)$. Therefore, under such simple condition of failure that the structure fails when the absolute value of the response reaches a critical value λ, the probability P_f that the structure will fail at any time is

$$P_f = P_x(\infty;\ -\lambda,\ \lambda) < 2\int_0^\infty h(\tau)\,d\tau \quad \cdots(10)$$

where, with the density function given in Eq. (9),

$$h(T) = \int_0^\infty \dot{x}\,\phi(\lambda,\dot{x};\ T)\,d\dot{x} = \int_{-\infty}^0 |\dot{x}|\,\phi(-\lambda,\dot{x};\ T)\,d\dot{x} \cdots\cdots(11)$$

By transformation $\dot{x}=\sigma_2[\eta(1-\rho^2)^{1/2}+\rho\lambda/\sigma_1]$,

$$\begin{aligned} h(T) &= \frac{\sigma_2}{2\pi\sigma_1}\exp\left[-\frac{1}{2}\left(\frac{\lambda}{\sigma_1}\right)^2\right] \\ &\quad \times\int_{\eta_1}^\infty [(1-\rho^2)^{1/2}\eta + \rho\lambda/\sigma_1]\exp\left(-\frac{\eta^2}{2}\right)d\eta \quad \cdots\cdots(12) \end{aligned}$$

where $\eta_1 = -\rho\lambda/\sigma_1(1-\rho^2)^{1/2}$.

Since it can be shown that

$$\int_{\eta_1}^\infty \eta\exp\left(-\frac{\eta^2}{2}\right)d\eta = \exp\left(-\frac{\eta_1^2}{1}\right)$$

and

$$\rho\int_{\eta_1}^\infty \exp\left(-\frac{\eta^2}{2}\right)d\eta < \delta\sqrt{2\pi\rho},$$

$$\begin{aligned} h(T) &< h^*(T) \\ &\equiv \frac{\sigma_2}{2\pi\sigma_1}\left[(1-\rho^2)^{1/2}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{\lambda}{\sigma_1}\right)^2\right\}\right. \\ &\quad \left.+\delta\sqrt{2\pi}\,\rho\frac{\lambda}{\sigma_1}\exp\left\{-\frac{1}{2}\left(\frac{\lambda}{\sigma_1}\right)^2\right\}\right] \quad \cdots\cdots(13) \end{aligned}$$

where

$$\delta=1 \text{ if } \rho>0,\ \delta=0 \text{ if } \rho\leq 0 \quad \cdots\cdots(14)$$

From Eqs. (10) and (13),

$$P_f = P_x(\infty;\ -\lambda,\ \lambda) < 2\int_0^\infty h^*(\tau)\,d\tau \cdots(15)$$

The last member of Eq. (15) with $h^*(\tau)$ of the form Eq. (13) is an upper bound of the probability of failure P_f of the structure subject to a Gaussian excitation.

As to the lower bound under the assumption that $x(t)$ is Gaussian and $\lambda_1=\lambda_2=\lambda$, one obtains from Eq. (1)

$$\begin{aligned} P_f = P_x(\infty;\ -\lambda,\lambda) &> P\{x(t_0)\leq -\lambda\} \\ &\quad + P\{x(t_0)\geq \lambda\} = 2\left[1-\Phi\left(\frac{\lambda}{\sigma_1^*}\right)\right] \quad \cdots\cdots(16) \end{aligned}$$

where σ_1^* is the maximum value of $\sigma_1(t)$ that occurs at $t=t_0$ and $\sqrt{2\pi}\,\Phi(x)=\int_{-\infty}^x \exp\left(-\frac{u^2}{2}\right)du$. The last member of Eq. (16) is a highest lower bound of the probability of failure P_f.

In the present investigation, Eqs. (15) and (16) are directly employed to evaluate the upper and lower bounds of the probability of failure of a simple structure subject to a Gaussian random excitation of earthquake type.

2. THE GROUND MOTION

General discussions concerning differentiation and integration involving random functions may be found in Reference 5.

In view of the evident condition that the ground velocity $\dot{f}(t)$ has to approach zero as time t increases to infinity, the following functions are assumed for the ground displacement $f(t)$, velocity $\dot{f}(t)$ and acceleration $\ddot{f}(t)$.

$$f(t)=\int_0^t (e^{-\alpha\tau}-e^{-\beta\tau})g(\tau)\,d\tau \quad\cdots\cdots(17)$$

$$\dot{f}(t)=(e^{-\alpha\tau}-e^{-\beta\tau})g(t) \quad\cdots\cdots(18)$$

$$\ddot{f}(t)=(-\alpha e^{-\alpha t}+\beta e^{-\beta t})g(t) + (e^{-\alpha t}-e^{-\beta t})\dot{g}(t) \quad\cdots\cdots(19)$$

where $\beta>\alpha>0$ and $g(t)$ is a stationary Gaussian random process with the mean

$$\langle g(t)\rangle=0 \quad\cdots\cdots(20)$$

and covariance

$$\langle g(t)g(s)\rangle=R(t-s) =\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)e^{i\omega(t-s)}\,d\omega \quad\cdots\cdots(21)$$

The square brackets indicate the ensemble average and $\Phi(\omega)=\Phi(-\omega)$ is the power spectrum of $g(t)$[6].

It is assumed that $\dot{g}(t)$ exists and is continuous in mean square; moreover, since $g(t)$ is stationary and Gaussian, so is $\dot{g}(t)$.

Hence, $\dot{f}(t)$ and $\ddot{f}(t)$ are Gaussian and continuous in mean square with mean

$$\langle\dot{f}(t)\rangle=\langle\ddot{f}(t)\rangle=0 \quad\cdots\cdots(22)$$

and covariance

$$\mu(t,s)=\langle\dot{f}(t)\dot{f}(s)\rangle =(e^{-\alpha t}-e^{-\beta t})(e^{-\alpha s}-e^{-\beta s})R(t-s) \quad\cdots\cdots(23)$$

$$\vartheta(t,s)=\langle\ddot{f}(t)\ddot{f}(s)\rangle=\frac{\partial^2\mu(t,s)}{\partial t\,\partial s} \quad\cdots\cdots(24)$$

$$\begin{aligned}&=(-\alpha e^{-\alpha t}+\beta e^{-\beta t})(-\alpha e^{-\alpha s}+\beta e^{-\beta s})\\ &\quad\times R(t-s)-(-\alpha e^{-\alpha t}-\beta e^{-\beta t})\\ &\quad\times(-e^{-\alpha s}+e^{-\beta s})\dot{R}(t-s)\\ &\quad+(e^{-\alpha t}-e^{-\beta t})(-\alpha e^{-\alpha s}+\beta e^{-\beta s})\dot{R}(t-s)\\ &\quad-(e^{-\alpha t}-e^{-\beta t})(e^{-\alpha s}-e^{-\beta s})\ddot{R}(t-s)\end{aligned} \quad\cdots\cdots(25)$$

where

$$\dot{R}(t-s)=\langle\dot{g}(t)g(s)\rangle=-\langle g(t)\dot{g}(s)\rangle \quad\cdots\cdots(26)$$

with $\dot{R}(0)=0$ because of assumed existence of the mean square derivative of $g(t)$ and

$$\ddot{R}(t-s)=-\langle\dot{g}(t)\dot{g}(s)\rangle \quad\cdots\cdots(27)$$

Since for $0<\tau,\ \tau'<t$

$$\left|\int_0^t\int_0^t R(\tau,\tau')\,d\tau\,d\tau'\right|\geq\left|\int_0^t\int_0^t \mu(\tau,\tau')\,d\tau\,d\tau'\right| \quad\cdots\cdots(28)$$

$f(t)$ in Eq. (17) exists in mean square under the assumption that the left hand side of (28) exists, and

$$\langle f(t)\rangle=0 \quad\cdots\cdots(29)$$

Also

$$\langle f^2(t)\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)[I_1(\omega,t)-I_2(\omega,t)] \times[\bar{I}_1(\omega,t)-\bar{I}_2(\omega,t)]\,d\omega \quad\cdots\cdots(30)$$

where

$$I_1(\omega,t)=\frac{1}{\alpha-i\omega}[1-e^{(-\alpha+i\omega)t}] \quad\cdots\cdots(31)$$

$$I_2(\omega,t)=\frac{1}{\beta-i\omega}[1-e^{(-\beta+i\omega)t}] \quad\cdots\cdots(32)$$

and $\bar{I}_k(\omega,t)\ (k=1,2)$ are the complex conjugates of $I_k(\omega,t)$. (This notation for complex conjugate is used throughout.)

$$\left.\begin{aligned}&\langle f'(0)\rangle=\langle\dot{f}^2(0)\rangle=0,\\ &\langle\ddot{f}^2(0)\rangle=(\alpha-\beta)^2R(0)\end{aligned}\right\} \quad\cdots\cdots(33)$$

$$\lim_{t\to\infty}\langle\dot{f}^2(t)\rangle=\lim_{t\to\infty}\langle\ddot{f}^2(t)\rangle=0 \quad\cdots\cdots(34)$$

$$\lim_{t\to\infty}\langle f^2(t)\rangle =\frac{(\alpha-\beta)^2}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)\frac{d\omega}{(\alpha^2+\omega^2)(\beta^2+\omega^2)} \quad\cdots\cdots(35)$$

The last equation provides the variance of the distribution of permanent displacement of the ground due to earthquake.

Available records from the major earthquakes in the past have not been analyzed to such an extent that the ground motion can be described in terms of particular forms of random process. Therefore, no claim is made that Eqs. (17), (18) and (19) are the random processes that describe the ground motion due to earthquake acceleration. These functions however represent possible forms of randomness which might be found in the ground motion, and are employed to illustrate the foregoing method of evaluating the probability of failure.

For later purposes, consider the following integral

$$I=\int_0^t\int_0^t G(\tau)H(\tau')\vartheta(\tau,\tau')\,d\tau\,d\tau' \quad\cdots\cdots(36)$$

where $G(\tau)$ and $H(\tau)$ are differentiable functions of τ in the interval $[0,t]$.

Integrating by parts and with the aid of Eq. (24),

$$I=G(t)H(t)\mu(t,t)-G(t)\int_0^t\dot{H}(\tau')\mu(t,\tau')\,d\tau'$$

$$-H(t)\int_0^t \dot{G}(\tau)\mu(\tau,t)d\tau$$
$$+\int_0^t\int_0^t \dot{G}(\tau)\dot{H}(\tau')\mu(\tau,\tau')d\tau\, d\tau' \quad \cdots\cdots(37)$$

3. MASS-SPRING SYSTEM WITH DAMPING

The structure considered here is a one-story building the mechanical model of which can be assumed to be mass supported by a spring coupled with a dashpot.

Hence, the equation of motion may be assumed to be

$$m\ddot{x}+\eta\dot{x}+kx=-m\ddot{f}(t) \quad \cdots\cdots(38)$$

where x is the displacement of the roof of mass m relative to the ground and k and η are shear stiffness and damping of the columns supporting the roof.

Under zero initial condition, the solution of Eq. (38) is

$$x(t)=-\int_0^t h(t-\tau)\ddot{f}(\tau)d\tau \cdots\cdots(39)$$

with

$$h(t)=\frac{1}{\omega_1}e^{-\mu t}\sin\omega_1 t \cdots\cdots(40)$$

where $2\mu=\frac{\eta}{m}$, $\omega_1=(\omega_0^2-\mu^2)^{1/2}$, $\omega_0^2=\frac{k}{m}$ and $0<\mu^2<\omega_0^2$.

Assuming that the integrals in Eq. (36) or (37) exist with $G(\tau)=h(t-\tau)$ and $H(\tau')=h(t-\tau')$ both differentiable and bounded in $(0,t)$, one can show that $x(t)$ in Eq. (39) exists in mean square. Then it can be shown that $x(t)$ possesses first and second derivatives in mean square. In particular

$$\dot{x}(t)=-\int_0^t \dot{h}(t-\tau)\ddot{f}(\tau)d\tau \cdots\cdots(41)$$

where

$$\dot{h}(t)=\frac{1}{\omega_1}\{\omega_1\cos\omega_1 t-\mu\sin\omega_1 t\}e^{-\mu t} \cdots\cdots(42)$$

Since $\ddot{f}(t)$ is Gaussian with mean zero, $x(t)$ and $\dot{x}(t)$ are also Gaussian with mean

$$\langle x(t)\rangle=\langle\dot{x}(t)\rangle=0 \cdots\cdots(43)$$

and, respectively, variance and covariance

$$\sigma_1^2(t)=\langle x^2(t)\rangle$$
$$=\frac{e^{-2\mu t}}{\omega_1^2}\int_0^t\int_0^t e^{\mu(\tau+\tau')}\sin\omega_1(t-\tau)\sin\omega_1(t-\tau')$$
$$\times\vartheta(\tau,\tau')d\tau\, d\tau' \quad \cdots\cdots(44)$$

$$\sigma_2^2(t)=\langle\dot{x}^2(t)\rangle=\frac{e^{-2\mu t}}{\omega_1^2}\int_0^t\int_0^t e^{\mu(\tau+\tau')}$$
$$\times\{\mu\sin\omega_1(t-\tau)-\omega_1\cos\omega_1(t-\tau)\}$$
$$\times\{\mu\sin\omega_1(t-\tau')-\omega_1\cos\omega_1(t-\tau')\}$$
$$\times\vartheta(\tau,\tau')d\tau\, d\tau' \quad \cdots\cdots(45)$$

$$\sigma_{12}(t)=\langle x(t)\dot{x}(t)\rangle$$
$$=\frac{e^{-2\mu t}}{\omega_1^2}\int_0^t\int_0^t e^{\mu(\tau+\tau')}\sin\omega_1(t-\tau)$$
$$\times\{\omega_1\cos\omega_1(t-\tau')-\mu\sin\omega_1(t-\tau)\}$$
$$\times\vartheta(\tau,\tau')d\tau\, d\tau' \quad \cdots\cdots(46)$$

Making use of Eq. (37), one obtains from Eqs. (44), (45) and (46)

$$\sigma_1^2(t)=\frac{e^{-2t\mu}}{\omega_1^2}\int_0^t\int_0^t$$
$$\times\{\mu\sin\omega_1(t-\tau)-\omega_1\cos\omega_1(t-\tau)\}e^{\mu\tau}$$
$$\times\{\mu\sin\omega_1(t-\tau')-\omega_1\cos\omega_1(t-\tau')\}$$
$$\times e^{\mu\tau'}\mu(\tau,\tau')d\tau\, d\tau' \quad \cdots\cdots(47)$$

$$\sigma_2^2(t)=\mu(t,t)+\frac{2e^{-\mu t}}{\omega_1}\int_0^t\{(\mu^2-\omega_1^2)\sin\omega_1(t-\tau')$$
$$-2\mu\omega_1\cos\omega_1(t-\tau')\}e^{\mu\tau'}\mu(t,\tau')d\tau'$$
$$+\frac{e^{-2\mu t}}{\omega_1^2}\int_0^t\int_0^t\{(\mu^2-\omega_1^2)\sin\omega_1(t-\tau)$$
$$-2\mu\omega_1\cos\omega_1(t-\tau)\}e^{\mu\tau}$$
$$\times\{(\mu^2-\omega_1^2)\sin\omega_1(t-\tau')$$
$$-2\mu\omega_1\cos\omega_1(t-\tau')\}e^{\mu\tau'}\mu(\tau,\tau')d\tau d\tau' \quad \cdots\cdots(48)$$

$$\sigma_{12}(t)=-\frac{e^{-\mu t}}{\omega_1}\int_0^t\{\mu\sin\omega_1(t-\tau)$$
$$-\omega_1\cos\omega_1(t-\tau)\}e^{\mu\tau}\mu(\tau,t)d\tau$$
$$-\frac{e^{-2\mu t}}{\omega_1^2}\int_0^t\int_0^t\{\mu\sin\omega_1(t-\tau)$$
$$-\omega_1\cos\omega_1(t-\tau)\}e^{\mu\tau}$$
$$\times\{(\mu^2-\omega_1^2)\sin\omega_1(t-\tau')$$
$$-2\mu\omega_1\cos\omega_1(t-\tau')\}e^{\mu\tau'}\mu(\tau,\tau')d\tau\, d\tau' \quad \cdots\cdots(49)$$

Furthermore, introducing Eq. (23) for $\mu(t-s)$ with $R(t-s)$ of the form of Eq. (21), one can show that Eqs. (47), (48) and (49) can be integrated with respect to τ and t' as follows :

$$\sigma_1^2(t)=[\mu^2K_1(t)+\omega_1^2K_2(t)$$
$$-\mu\omega_1\{K_3(t)+K_4(t)\}]/\omega_1^2 \quad \cdots\cdots(50)$$

$$\sigma_2^2(t)=[\omega_1^2(e^{-\alpha t}-e^{-\beta t})^2K_0+(\mu^2-\omega_1^2)^2K_1(t)$$
$$+4\mu^2\omega_1^2K_2(t)-2\mu\omega_1(\mu^2-\omega_1^2)$$
$$\times\{K_3(t)+K_4(t)\}$$
$$+2\omega_1\{(\mu^2-\omega_1^2)(e^{-\alpha t}-e^{-\beta t})K_5(t)$$
$$-2\mu\omega_1(e^{-\alpha t}-e^{-\beta t})K_6(t)\}]/\omega_1^2 \quad \cdots(51)$$

$$\sigma_{12}(t)=[-\mu(\mu^2-\omega_1^2)K_1(t)-2\mu\omega_1^2K_2(t)$$
$$+2\mu^2\omega_1K_3(t)+\omega_1(\mu^2-\omega_1^2)K_4(t)$$
$$-\omega_1\mu(e^{-\alpha t}-e^{-\beta t})\bar{K}_s(t)$$

$$+\omega_1^2(e^{-\alpha t}-e^{-\beta t})\bar{K}_6(t)]/\omega_1^2 \cdots\cdots (52)$$

where

$$K_0=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)d\omega \cdots\cdots (53)$$

$$K_1(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)J_1(\omega,t)\bar{J}_1(\omega,t)d\omega \cdots\cdots (54)$$

$$K_2(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)J_2(\omega,t)\bar{J}_2(\omega,t)d\omega \cdots\cdots (55)$$

$$K_3(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)J_1(\omega,t)\bar{J}_2(\omega,t)d\omega \cdots\cdots (56)$$

$$K_4(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)\bar{J}_1(\omega,t)J_2(\omega,t)d\omega \cdots\cdots (57)$$

$$K_5(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)e^{i\omega t}\bar{J}_1(\omega,t)d\omega \cdots (58)$$

$$K_6(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)e^{i\omega t}\bar{J}_2(\omega,t)d\omega \cdots (59)$$

with

$$J_1(\omega,t)=e^{-\mu t}\int_0^t[e^{(\mu-\alpha+i\omega)\tau}-e^{(\mu-\beta+i\omega\tau)}] \times \sin\omega_1(t-\tau)d\tau \cdots\cdots (60)$$

$$J_2(\omega,t)=e^{-\mu t}\int_0^t[e^{(\mu-\alpha+i\omega)\tau}-e^{(\mu-\beta+i\omega\tau)}] \times \cos\omega_1(t-\tau)d\tau \cdots\cdots (61)$$

Let $G_k(\omega, t; \theta)$ and $H_k(\omega, t; \theta)$ ($k=1$, 2 and $\theta=\alpha$ or β) be defined as follows :

$$\begin{aligned}G_1(\omega, t; \theta)&=R_e e^{-\mu t}\int_0^t e^{(\mu-\theta+i\omega)}\sin\omega_1(t-\tau)d\tau\\&=\{(\mu-\theta)^2+\omega_1^2-\omega^2\}\{e^{-\theta t}\cos\omega t\\&\quad -e^{-\mu t}(\mu-\theta)\sin\omega_1 t-e^{-\mu t}\cos\omega_1 t\}\\&\quad +2(\mu-\theta)\omega\{e^{-\theta t}\sin\omega t\\&\quad -e^{-\mu t}\omega\sin\omega_1 t\}]/Y(\omega,\theta) \cdots (62)\end{aligned}$$

$$\begin{aligned}H_1(\omega, t; \theta)&=\vartheta_m e^{-\mu t}\int_0^t e^{(\mu-\theta+i\omega)}\sin\omega_1(t-\tau)d\tau\\&=[\{(\mu-\theta)^2+\omega_1^2-\omega^2\}\{e^{-\theta t}\sin\omega t\\&\quad -e^{-\mu t}\omega\sin\omega_1 t\}-2(\mu-\theta)\omega\\&\quad\times\{e^{-\theta t}\cos\omega t-e^{-\mu t}(\mu-\theta)\sin\omega_1 t\\&\quad -e^{-\mu t}\cos\omega_1 t\}]/Y(\omega,\theta) \cdots\cdots (63)\end{aligned}$$

$$\begin{aligned}G_2(\omega, t; \theta)&=R_e e^{-\mu t}\int_0^t e^{(\mu-\theta+i\omega)}\cos\omega_1(t-\tau)d\tau\\&=[\{(\mu-\theta)^2+\omega_1^2-\omega^2\}\\&\quad\times\{e^{-\theta t}(\mu-\theta)\cos\omega t-e^{-\theta t}\omega\sin\omega t\\&\quad -e^{-\mu t}(\mu-\theta)\cos\omega_1 t+e^{-\mu t}\sin\omega_1 t\}\\&\quad +2(\mu-\theta)\omega\{e^{-\theta t}\omega\cos\omega t\\&\quad +e^{-\theta t}(\mu-\theta)\sin\omega t\\&\quad -e^{-\mu t}\omega\cos\omega_1 t\}]/Y(\omega,\theta) \cdots (64)\end{aligned}$$

$$\begin{aligned}H_2(\omega, t; \theta)&=\vartheta_m e^{-\mu t}\int_0^t e^{(\mu-\theta+i\omega)}\cos\omega_1(t-\tau)d\tau\\&=[\{(\mu-\theta)^2+\omega_1^2-\omega^2\}\\&\quad\times\{e^{-\theta t}\omega\cos\omega t+e^{-\theta t}(\mu-\theta)\sin\omega t\\&\quad -e^{-\mu t}\omega\cos\omega_1 t\}-2(\mu-\theta)\omega\\&\quad\times\{e^{-\theta t}(\mu-\theta)\cos\omega t-e^{-\theta t}\omega\sin\omega t\\&\quad -e^{-\mu t}(\mu-\theta)\cos\omega_1 t\\&\quad +e^{-\mu t}\sin\omega_1 t\}]/Y(\omega,\theta) \cdots\cdots (65)\end{aligned}$$

with

$$Y(\omega,\theta)=\{(\mu-\theta)^2+\omega_1^2-\omega^2\}^2+4(\mu-\theta)^2\omega^2 \cdots\cdots (66)$$

where $R_e z$ and $\vartheta_m z$ denote the real and imaginary part of a complex number z.

Hence, Eqs. (60) and (61) can be written as

$$\begin{aligned}J_k(\omega,t)&=G_k(\omega, t; \alpha)-G_k(\omega, t; \beta)\\&\quad +i\{H_k(\omega, t; \alpha)-H_k(\omega, t; \beta)\}\\&(k=1,2) \cdots\cdots (67)\end{aligned}$$

With the nondimensional time t^*, frequency ξ, damping μ'

$$t_*=\omega_1 t,\quad \xi=\omega/\omega_1,\quad \mu'=\mu/\omega_1 \cdots\cdots (68)$$

and also with θ' representing

$$\alpha'=\alpha/\omega_1,\ \text{or}\ \beta'=\beta/\omega_1 \cdots\cdots (69)$$

one can derive the non-dimensional variances and correlation function of $x(t)$ and $\dot{x}(t)$ as follows :

$$F_1(t_*)=\frac{\omega_1\sigma_1^2(t^*/\omega_1)}{D}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi^*(\omega_1\xi)Z_1(\xi,t^*)d\xi \cdots\cdots (70)$$

$$F_2(t^*)=\frac{\sigma_2^2(t^*/\omega_1)}{\omega_1 D}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi^*(\omega_1\xi)Z_2(\xi,t^*)d\xi \cdots\cdots (71)$$

$$F_3(t^*)=\frac{\sigma_{12}(t^*/\omega_1)}{D}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi^*(\omega_1\xi)Z_3(\xi,t^*)d\xi \cdots\cdots (72)$$

where D is a constant with the dimension of length²/time and

$$\Phi^*(\omega_1\xi)=\Phi(\omega)/D \cdots\cdots (73)$$

$$\begin{aligned}Z_1(\xi,t^*)&=\mu'^2|J_1(\xi,t^*)|^2+|J_2(\xi,t^*)|^2\\&\quad -2\mu' R_e\{J_1(\xi,t^*)\bar{J}_2(\xi,t^*)\} \cdots (74)\end{aligned}$$

$$\begin{aligned}Z_2(\xi,t^*)&=(e^{-\alpha' t^*}-e^{-\beta' t^*})^2+2(e^{-\alpha' t^*}-e^{-\beta' t^*})\\&\quad\times[(\mu'^2-1)R_e\{e^{i\xi t^*}\bar{J}_1(\xi,t^*)\}\\&\quad -2\mu' R_e\{e^{i\xi t^*}\bar{J}_2(\xi,t^*)\}]\end{aligned}$$

$$+(\mu'^2-1)^2|J_1(\xi,t^*)|^2$$
$$+4\mu'^2|J_2(\xi,t^*)|^2$$
$$-4\mu'(\mu'^2-1)R_e\{\bar{J}_1(\xi,t^*)\bar{J}_2(\xi,t^*)\} \quad \cdots\cdots(75)$$

$$Z_3(\xi,t^*)=-(e^{-\alpha't^*}-e^{-\beta't^*})$$
$$\times[\mu'R_e\{e^{i\xi t^*}\bar{J}_1(\xi,t^*)\}$$
$$-R_e\{e^{i\xi t^*}\bar{J}_2(\xi,t^*)\}]$$
$$-[\mu'(\mu'^2-1)|J_1(\xi,t^*)|^2$$
$$-(3\mu'^2-1)R_e\{J_1(\xi,t^*)\bar{J}_2(\xi,t^*)\}$$
$$+2\mu'|J_2(\xi,t^*)|^2] \quad \cdots\cdots(76)$$

In Eqs. (75), (76) and (77),

$J_k(\xi,t^*)\equiv J_k(\omega,t)$ with replacement of $\omega_1,\omega,t,\mu,\alpha$ and β by $1,\xi,t^*,\mu'$, α' and β' respectively $(k=1,2)$ $\cdots\cdots(77)$

or

$$J_k(\xi,t^*)=G_k(\xi,t^*,\alpha')-G_k(\xi,t^*;\beta')$$
$$+i\{H_k(\xi,t^*;\alpha')-H_k(\xi,t^*;\beta')\}$$
$$(k=1,2) \quad \cdots\cdots(78)$$

where

$$\left.\begin{array}{l} G_k(\xi,t^*;\theta')\equiv G_k(\omega,t;\theta) \\ H_k(\xi,t^*;\theta')\equiv H_k(\omega,t;\theta) \end{array}\right\}$$

with replacement of ω_1,ω,t,μ and θ by $1,\xi,t^*,\mu'$ and θ' $(k=1,2)$ $\cdots\cdots(79)$

and therefore,

$$|J_k(\xi,t^*)|^2=\{G_k(\xi,t^*,\alpha')-G_k(\xi,t^*;\beta')\}^2$$
$$+\{H_k(\xi,t^*;\alpha')-G_k(\xi,t^*;\beta'\}^2$$
$$(k=1,2) \quad \cdots\cdots(80)$$

$$R_e\{J_1(\xi,t^*)\bar{J}_2(\xi,t^*)\}$$
$$=\{G_1(\xi,t^*;\alpha')-G_1(\xi,t^*;\beta')\}\{G_2(\xi,t^*;\alpha')$$
$$-G_2(\xi,t^*;\beta')\}+\{H_1(\xi,t^*;\alpha')$$
$$-H_1(\xi,t^*;\beta')\}\{H_2(\xi,t^*;\alpha')$$
$$-H_2(\xi,t^*;\beta')\} \quad \cdots\cdots(81)$$

$$R_e\{e^{i\xi t}\bar{J}_k(\xi,t^*)\}$$
$$=[G_R(\xi,t^*;\alpha')-G_R(\xi,t^*;\beta')]\cos\xi t^*$$
$$+[H_k(\xi,t^*;\alpha')-H_k(\xi,t^*;\beta')]\sin\xi t^*$$
$$(k=1,2) \quad \cdots\cdots(82)$$

Since it is assumed that the failure of the structure will occur when the absolute value of x exceeds a critical value λ, Eq. (15) with Eq. (13) and Eq. (16) can be used to find the upper and lower bound of the probability of failure P_f.

When Eq. (13) with $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_{12}(t)$ given in Eqs. (50), (51) and (52) is substituted, Eq. (15) becomes

$$P_f<\frac{1}{\pi}\int_0^\infty\sqrt{F_2/F_1}\Big[(1-\rho^2)^{1/2}\exp$$
$$\times\left\{-\frac{1}{2(1-\rho^2)}(k_0^2F_1^*/F_1)\right\}$$
$$+\delta\sqrt{2\pi}\,\rho\,k_0F_1^*/F_1\exp$$
$$\times\left\{-\frac{1}{2}(k_0^2F_1^*/F_1)\right\}\Big] \quad \cdots\cdots(83)$$

where $\delta=1$ for $\rho>0$, $\delta=0$ for $\rho\leqslant0$, k_0 is the non-dimensional location of the barrier λ in terms of the maximum standard deviation $\sigma_1^*=\sqrt{\frac{DF_1^*}{\omega_1}}=\sqrt{\frac{DF_1(t_0)}{\omega_1}}$ of the displacement $x(t)$ occurring at $t=t_0$, $F_1=F_1(t)$, $F_2=F_2(t)$, $F_1^*=F_1(t_0)$

$$\rho=\rho(t^*)=F_3(t^*)/\sqrt{F_1(t^*)F_2(t^*)} \quad \cdots(84)$$

The lower bound can be easily obtained from Eq. (16) where $\lambda/\sigma_1^*=k_0$.

4. NUMERICAL EVALUATION OF THE BOUNDS

Purely for the purpose of numerical computation, $\Phi(\omega)$ is assumed to be

$$\Phi(\omega)=De^{-a^2\omega^2}=De^{-a^2\omega_1^2\xi^2} \quad \cdots\cdots(85)$$

with which the correlation function $R(\tau)=(D/2\sqrt{\pi}\,a)\exp\left(-\frac{\tau^2}{4a^2}\right)$ is associated.

Then, at various values of $t^*(=\Delta t^*, 2\Delta t^*,\cdots)$, one can evaluate without much difficulty $F_1(t^*)$, $F_2(t^*)$ and $F_3(t^*)$ from Eqs. (70), (71) and (72) performing the integrations with respect to ξ using an electronic digital computer IBM 7094. The numerical example is given for the case where $\mu'=0.204$, $\alpha'=0.408$, $\beta'=0.816$ and $a^2\omega_1^2=0.475$ with $\Delta t^*=0.1$. These values correspond to for example, $\omega_0=10\,\pi$/sec, $\mu=2\,\pi$/sec, $\alpha=4\,\pi$/sec, $\beta=8\,\pi$/sec and $a^2=0.0005$ sec^2. $F_1(t^*)$, $F_2(t^*)$ and $\rho(t^*)$ for this set of parameters are shown in Fig. 1 from which $\sqrt{F_1^*}$ is found to be 0.146.

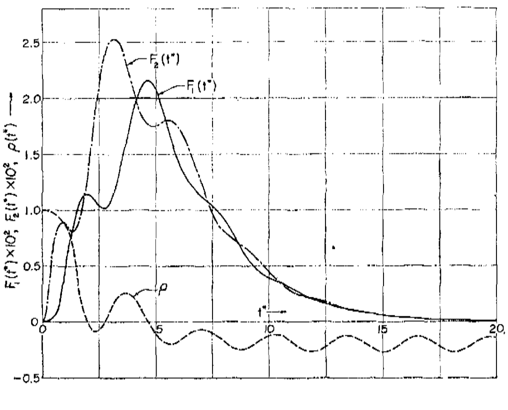

Fig. 1 Nondimensional Variances $F_1(t^*)$ and $F_2(t^*)$ and Correlation Function $\rho(t^*)$ of Displacement and Velocity.

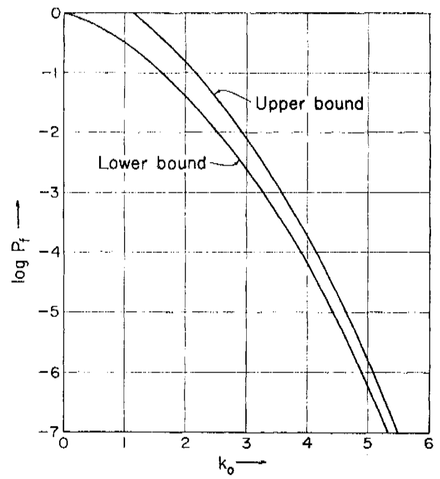

Fig. 2 Upper and Lower Bounds of Probability of Failure.

Evaluation $F_1(t^*)$, $F_2(t^*)$ and $F_3(t^*)$ at $t^* = \Delta t^*$, $2\Delta t^*$, $3\Delta t^*$, etc. makes it possible to use Eq. (83) for the upper bound for various values of k_0 again with the aid of IBM 7094. Although it is not discussed here, the convergence of the infinite integral in Eq. (83) with $F_1(t)$, $F_2(t)$ and $F_3(t)$ in Eqs. (70), (71) and (72) can be proved.

As to the lower bound, one can immediately use Eq. (16) as mentioned before.

The bounds are shown in Fig. 2 which indicates that the ratio of the upper bound to the lower one is approximately 3.0 over the range of the probability considered $(1.0 \sim 10^{-7})$.

Hence, the present method seems useful at least for order estimation of the probability of failure of a simple structure subject to random excitation of earthquake type in which the excitation dies down rapidly.

5. CONCLUSION AND ACKNOWLEDGEMENT

A method has been presented to find upper and lower bounds of the probability that a simple structure will fail under a random loading of earthquake type. With the aid of a digital computer IBM 7090, a numerical example has been given when the failure is assumed to occur as soon as the absolute value of the displacement of the structure exceeds a critical value. The example has shown that the present method produces the upper and lower bounds close enough at least for the order estimation of the probability.

This work was done under the auspices of the Institute for the study of Fatigue and Reliability supported jointly by the Office of Naval Research, Air Force Materials Laboratory and Advanced Research Project Agency under contract Nonr 266(91).

REFERENCES

1) J.L. Bogdanoff, J.E. Goldberg and M.C. Bernard; "Response of a Simple Structure to a Random Earthquake-Type Disturbance", Bulletin of the Seismological Society of America, Vol. 91, 1961, pp. 293-310.
2) V.V. Bolotin; "Application of Statistical Methods to Estimation of Safety of Structures Subject to Seismic Influences", Institute of Mechanics of Academy of Science, USSR, Engineering Journal, Vol. XXVII, 1960, pp. 56-68, (in Russian).
3) M. Shinozuka; "On the Two-sided Barrier Problem", to appear in the Journal of Applied Probability, Vol. 2, No. 2, June, 1965.
4) S.O. Rice; "Mathematical Analysis of Random Noise", in "Selected Papers on Noise and Stochastic Processes," edited by N. Wax, Dover Publicatlon, New York, 1954, p. 190.
5) J.E. Moyal; "Statistical Processes and Statistical Physics", Jour. Roy. Stat. Soc., Ser. B, Vol. 11, 1949, p. 168.
6) V.V. Solodovnikov; "Introduction to the Statistical Dynamics of Automatic Control Systems", English translation edited by J.B. Thomas and L.A. Zadeh, Dover Publications, Inc., New York, 1960, p. 98.